**Kuwait University**
College of Engineering and Petroleum

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# ME417 CONTROL OF MECHANICAL SYSTEMS

## PART I: INTRODUCTION TO FEEDBACK CONTROL

## LECTURE 3: LAPLACE TRANSFER FUNCTION

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- Objectives:
  - *Review The Laplace Transform*
  - *Review The Inverse Laplace Transform and Partial Fraction Expansion*
  - *Introduce Transfer Functions of Mechanical Systems*
- Reading:
  - *Nise: 2.1-2.3, 2.5.-2.6*
- *Practice Problems*

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- The Laplace Transform is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Where $s = \sigma + j\omega$

- The **Inverse** Laplace Transform is defined as

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)$$

Where $u(t)$ is the unit step function:

$$u(t) = 1 \quad t > 0$$
$$u(t) = 0 \quad t < 0$$

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# Laplace Transform Table – Table 2.1

| Item no. | $f(t)$ | $F(s)$ |
|---|---|---|
| **1.** | $\delta(t)$ | $1$ |
| **2.** | $u(t)$ | $\frac{1}{s}$ |
| **3.** | $tu(t)$ | $\frac{1}{s^2}$ |
| **4.** | $t^2u(t)$ | $\frac{1}{s^n+1}$ |
| **5.** | $e^{-at}u(t)$ | $\frac{1}{s+a}$ |
| **6.** | $sin\omega t\, u(t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| **7.** | $cos\omega t\, u(t)$ | $\frac{s}{s^2+\omega^2}$ |

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# Laplace Transform Theorems – Table 2.2

| Item no. | Theorem | Name |
|---|---|---|
| 1. | $\mathcal{L}[f(t)] = F(s) = \int_{0_-}^{\infty} f(t)e^{-st}dt$ | Definition |
| 2. | $\mathcal{L}[kf(t)] = kF(s)$ | Linearity Theorem |
| 3. | $\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$ | Linearity Theorem |
| 4. | $\mathcal{L}[e^{-at}f(t)] = F(s+a)$ | Frequency Shift Theorem |
| 5. | $\mathcal{L}[f(t-T)] = e^{-sT}F(s)$ | Time Shift Theorem |
| 6. | $\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$ | Scaling Theorem |

# Laplace Transform Theorems – Table 2.2

| Item no. | Theorem | Name |
|---|---|---|
| **7.** | $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0_-)$ | Differentiation Theorem |
| **8.** | $\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf''(0_-) - f'(0_-)$ | Differentiation Theorem |
| **9.** | $\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0_-)$ | Differentiation Theorem |
| **10.** | $\mathcal{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$ | Integration Theorem |
| **11.** | $f(\infty) = \lim_{s\to 0} sF(s)$ | Final Value Theorem |
| **12.** | $f(0+) = \lim_{s\to\infty} sF(s)$ | Initial Value Theorem |

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- To find the inverse Laplace of a complicated function, we can convert to a sum of multiple terms, using partial fraction expansion

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0}$$

$$F(s) = \frac{N(s)}{(s+p_n)(s+p_{n-1}) \ldots (s+p_1)} = \frac{K_n}{(s+p_n)} + \frac{K_{n-1}}{(s+p_{n-1})} + \cdots + \frac{K_1}{(s+p_1)}$$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{K_n}{(s+p_n)}\right] + \mathcal{L}^{-1}\left[\frac{K_{n-1}}{(s+p_{n-1})}\right] + \cdots + \mathcal{L}^{-1}\left[\frac{K_1}{(s+p_1)}\right]$$

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# PFE – Case 1: D(s) Has real and distinct Roots

$$F(s) = \frac{N(s)}{D(s)} = \frac{2}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

To find $K_1$, we multiply the above equation by $(s+1)$

$$\frac{2}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)}$$

Letting $s = -1$, eliminates the right term and gives $K_1 = 2$.

Repeat the process to get $K_2 = -2$

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$$F(s) = \frac{N(s)}{D(s)} = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{s+1} + \frac{K_2}{(s+2)^2} + \frac{K_3}{s+2}$$

To get $K_1$, multiply by $(s+1)$ and set $s = -1$.
To get $K_2$, multiply by $(s+2)^2$ and set $s = -2$

$$\frac{2}{(s+1)} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3$$

To get $K_3$, first differentiate the above and set $s = -2$

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2} K_1 + K_3$$

Gives $K_3 = -2$

$$F(s) = \frac{N(s)}{D(s)} = \frac{3}{s(s^2+2s+5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2s+5}$$

$K_1$ is found by multiplying by $s$, setting $s = 0$, giving $K_1 = \frac{3}{5}$

To find $K_2, K_3$, multiply by the least common denominator $s(s^2 + 2s + 5)$, and simplify

$$3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3$$

Solve for $\left(K_2 + \frac{3}{5}\right) = 0$, $\left(K_3 + \frac{6}{5}\right) = 0$, gives $K_2 = -\frac{3}{5}$ and $K_3 = -\frac{6}{5}$

$$F(s) = \frac{3}{5s} - \frac{3(s+2)}{5(s^2+2s+5)}$$

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- When modeling a dynamic system, we get a differential equation.
- For linear time-invariant, single-input single-output systems:

$$\frac{d^m c(t)}{dt^n} + d_{n-1}\frac{d^{m-1}c(t)}{dt^{n-1}} + \cdots + d_0 c(t) = b_m \frac{b^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1}c(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

Where $r(t)$ is the input and $c(t)$ is the output

- Taking the Laplace transfer of both sides

$$a_n s^n C(s) + a_{n-1}s^{n-1}C(s) + \cdots + a_0 C(s) + initial\ condition\ terms\ involving\ c(t) =$$
$$b_m s^m R(s) + b_{m-1}s^{m-1}R(s) + \cdots + b_0 R(s) + initial\ condition\ terms\ involving\ r(t)$$

- Assuming zero initial conditions gives

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}$$

- The transfer function is thus defined as:

*The algebraic relationship between the output to the input of a* ***linear****,* ***time-invariant****,* ***single-input single-output*** *system, in the* ***Laplace domain*** *assuming* ***zero initial conditions***.

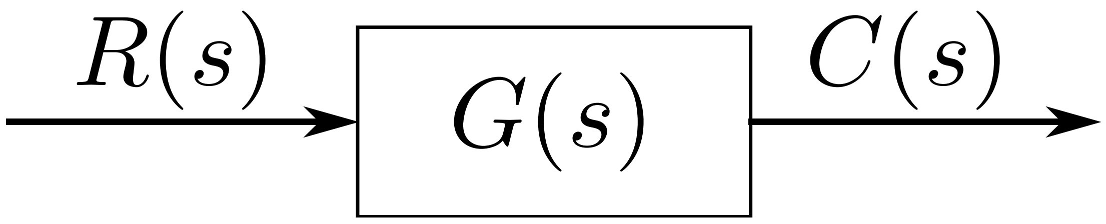
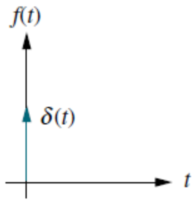
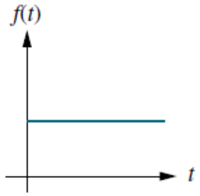
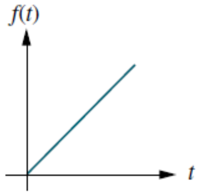
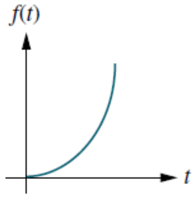
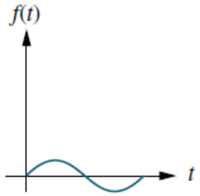


**TABLE 1.1** Test waveforms used in control systems

| Input | Function | Description | Sketch | Use |
|---|---|---|---|---|
| Impulse | $\delta(t)$ | $\delta(t) = \infty$ for $0- < t < 0+$<br>$= 0$ elsewhere<br>$\int_{0-}^{0+} \delta(t)dt = 1$ |  | Transient response<br>Modeling |
| Step | $u(t)$ | $u(t) = 1$ for $t > 0$<br>$= 0$ for $t < 0$ |  | Transient response<br>Steady-state error |
| Ramp | $tu(t)$ | $tu(t) = t$ for $t \geq 0$<br>$= 0$ elsewhere |  | Steady-state error |
| Parabola | $\frac{1}{2}t^2u(t)$ | $\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$<br>$= 0$ elsewhere |  | Steady-state error |
| Sinusoid | $\sin \omega t$ | |  | Transient response<br>Modeling<br>Steady-state error |

Given the following differential equation, find the time response equation to a step input

$$\frac{d^2c}{dt^2} + 12\frac{dc}{dt} + 36c = \frac{dr}{dt} + 3r$$

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Find the transfer function corresponding to the following differential equation, then perform a partial fraction expansion and retrieve the time response to a unit step input.

# Example 2

$$\frac{d^2c}{dt^2} + 6\frac{dc}{dt} + 18c = 3r$$

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## Example 3

Find the ramp response for a system whose transfer function is

$$G(s) = \frac{s+1}{(s+3)^2}$$

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# Practice Problem 1

Using the Laplace transform pairs of Table 2.1 and the Laplace transform theorems of Table 2.2, derive the Laplace transforms for the following time functions: [Section: 2.2]

a. $e^{-at} sin\omega t\, u(t)$

b. $e^{-at} cis\omega t\, u(t)$

c. $t^3\, u(t)$

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Practice Problem 2

Find the expression for the transfer function of the systems given by the following differential equations

a. $$\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + y = \frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 7x$$

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## Practice Problem

Find the time function corresponding to the following Laplace transforms.
Hint: You can verify your partial fraction expansion using MATLAB's residue() function.

a. $G(s) = \frac{1}{s(s+2)^2}$

b. $G(s) = \frac{2(s^2+s+1)}{s(s+1)^2}$

c. $G(s) = \frac{(s^2-1)}{(s^2+1)^2}$

d. $G(s) = \frac{7}{s^2(s+11)(s+12)}$

e. $G(s) = \frac{1}{s(s+2)^2}$

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