Kuwait University
College of Engineering and Petroleum

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ME417 CONTROL OF MECHANICAL SYSTEMS

Part II: Controller Design Using Root-Locus

Lecture 1: Introduction to Root-Locus

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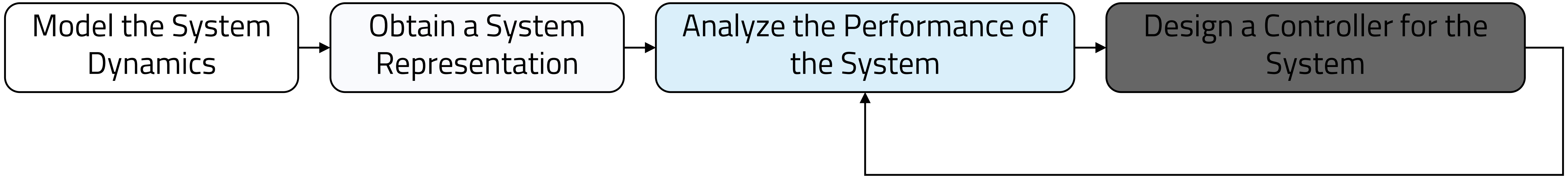
Lecture Plan

- Objectives:
  - Review the anatomy of a control system block diagram
  - Introduction to the concept of root-locus diagrams
  - Overview the properties of the root-locus
- Reading:
  - Nise: 8.1-8.3
- Practice problems are more applicable after the subsequent lectures.

- We will now begin to introduce the first main technique of designing a controller.
- The design technique we will learn in this part is a graphical technique, it offers an alternative and qualitative method to understand the behavior of a dynamic system.
- It is also considered a way to represent the system and a method to analyze the performance (inherent to the design intent).



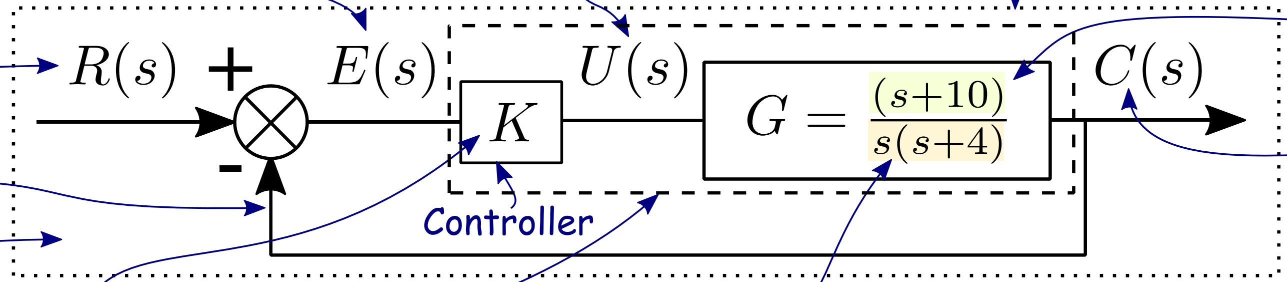
Feedback Error

Controller Output: Input to Plant

Closed-Loop Form of Feedback System

Reference Input

Gives Open-Loop Zeros



Feedback Signal

Controller

Closed-Loop System Output

Closed-Loop System: Feedback

Controller Design Variable

Open-Loop Characteristic Polynomial

$s(s+4)=0 \Rightarrow s=0,-4$ ← Gives Open-Loop Poles

Open-Loop Transfer Function
a.k.a Forward Transfer Function: $G_{ol} = K\frac{(s+10)}{s(s+4)}$

via Block Diagram Reduction we convert the Closed-Loop **form** of a Feedback System, to an equivalent Open-Loop **form**

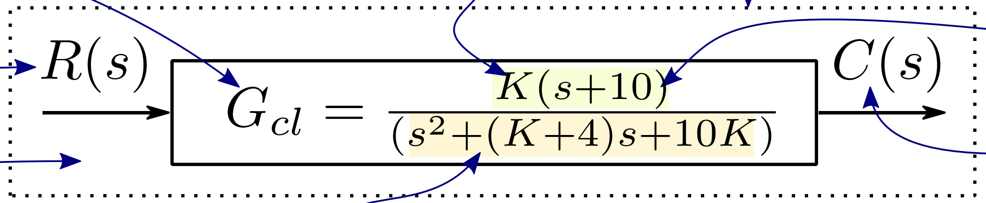
Closed-Loop Transfer Function

Controller Design Variable

Equivalent Form of Feedback System

Reference Input

Gives Closed-Loop Zeros: Same as Open-Loop Zeros



Closed-Loop System: Feedback

Closed-Loop System Output

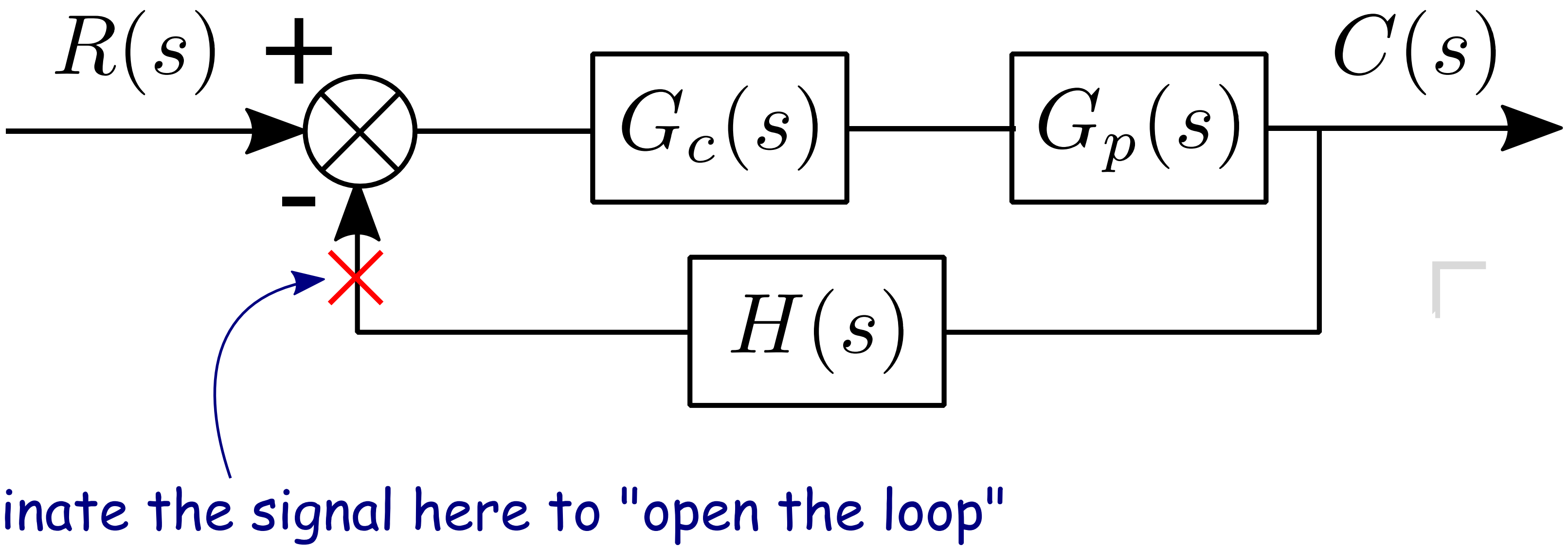
Closed-Loop Characteristic Polynomial

$1+G_{ol} = s^2+(4+K)s+10K = 0,\ s = -\frac{K+4}{2} \pm \frac{\sqrt{(K+4)^2-40K}}{2}$ ← Gives Closed-Loop Poles: A function of the gain K

Open-Loop Transfer Function of a Non-Unity Feedback System

- The open-loop transfer function of a ***any*** feedback system is obtained by terminating the feedback signal at just before the summation block. And multiplying all the blocks in series up to the termination point.
- For a Non-Unity Feedback System, the open-loop transfer function is:

$$G_{ol} = G_C(s)G_P(s)H(s)$$



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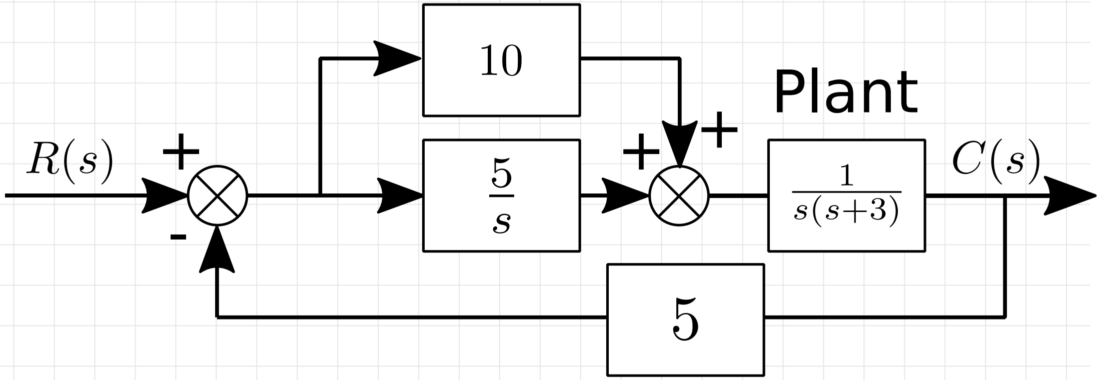
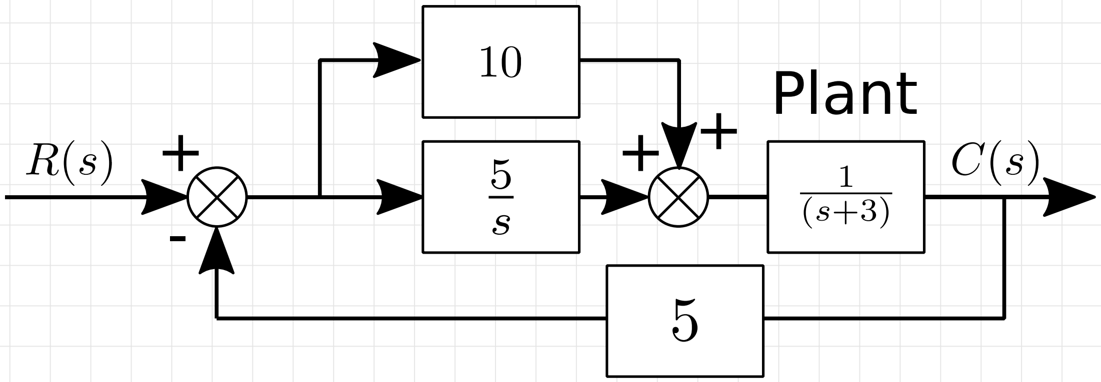
For the following unity feedback system. Derive the following:

a. The plant transfer function
b. The controller transfer function
c. The open-loop transfer function without feedback
    - (The open-loop **system** T.F.)
d. The open-loop transfer function with feedback
e. The forward transfer function
f. The closed-loop transfer function
g. The closed-loop characteristic polynomial
h. The input to the plant in terms of the reference.
i. The controller design variables

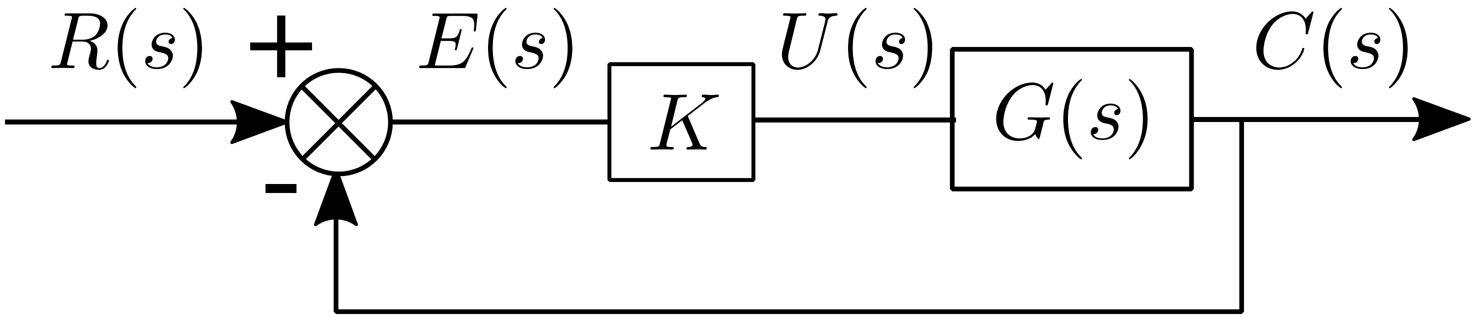


For the following unity feedback system, place the:

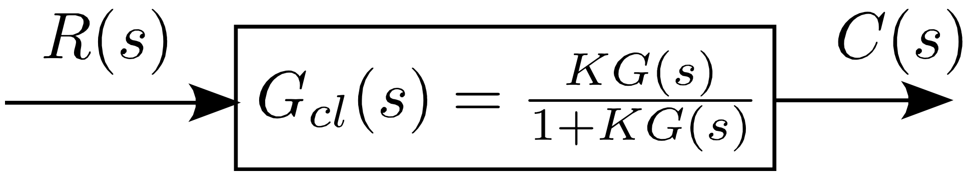
a. Plant poles and zeros
b. Open-loop poles and zeros
c. Closed-loop poles and zeros



- Given a feedback control system of the form

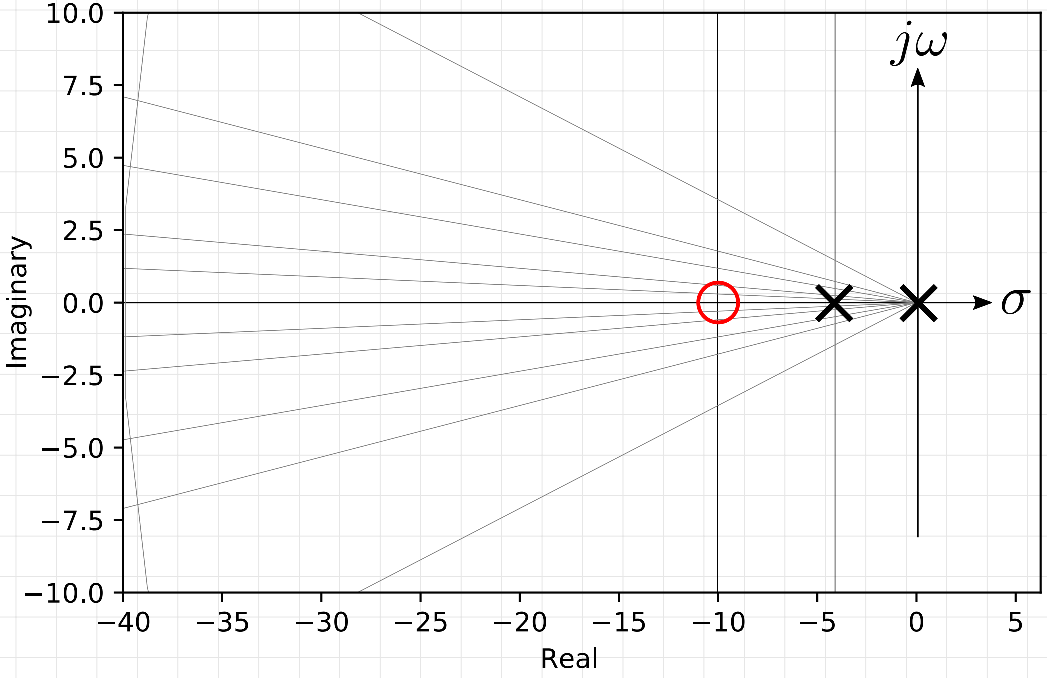


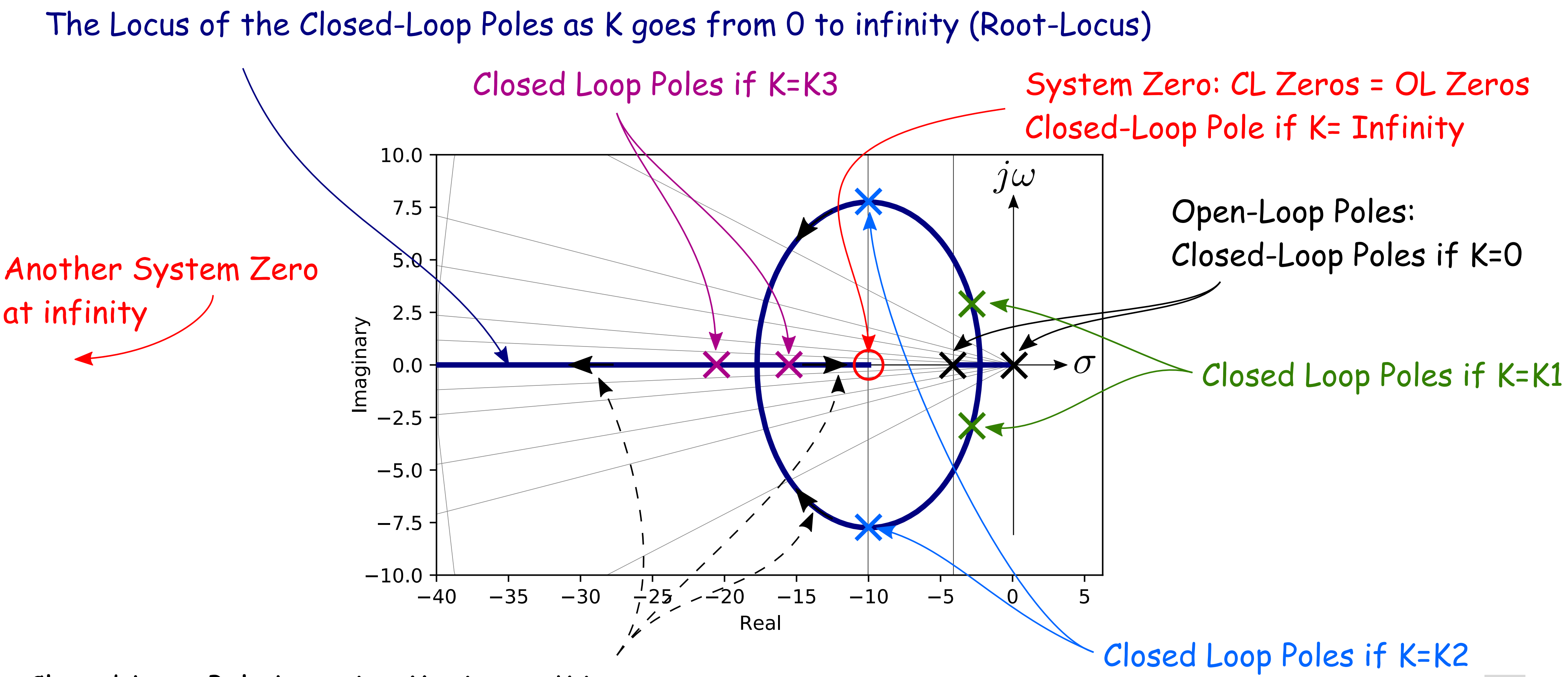
  - With the equivalent open-loop **form**:



- The root-locus is the locus of the roots of the characteristic polynomial of the closed-loop transfer function: $1 + KG_{ol} = 0$, on the s-plane, as K goes from 0 to ∞

$$G_{ol} = K\frac{(s+10)}{s(s+4)}$$





Closed-Loop Pole Location Motion as K increases:

As K increases, CL Poles move away from OL poles and toward system zeros:

Here: K1<K2<K3

$$G_{ol} = K\frac{(s+10)}{s(s+4)} \qquad G_{cl} = \frac{K(s+10)}{(s^2+(K+4)s+10K)}$$

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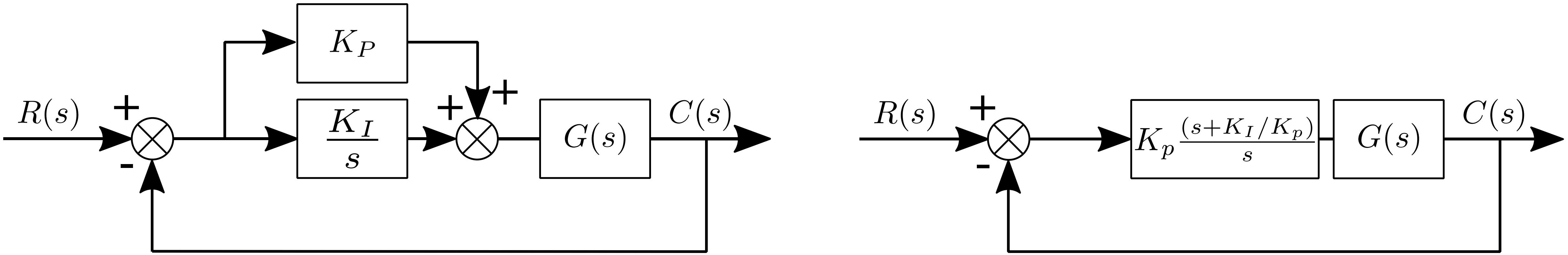
What if we have more than just one gain?

- The root locus is drawn for a single variable K, what if there is more than one gain in the feedback loop, as is the case when applying a PID controller?
- Take the **PI** Controller:

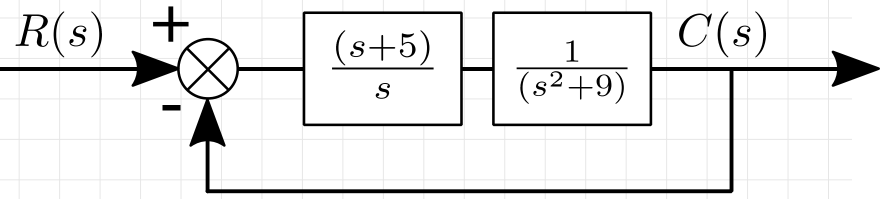


- We can factor out the proportional gain K_p as the root locus variable and fix the ratio K_I/K_p. The ratio $K_I/K_p = z_1$, determines the location of the added zero by the PI controller. The root-locus will become a variable of K_p
- Char. Poly.: $1 + K_p \frac{(s+z_1)G(s)}{s} = 0$

We wish to apply a PI feedback controller on a dc motor to control its speed. Find the values of the gain K_I, if $K_p = 1$, to achieve a settling time of $0.25s$, when a step input of 10rad/s is applied.

Example

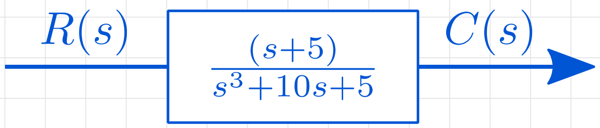
Given the following unity feedback system, determine the location of the open-loop poles, then compute the characteristic polynomial for the closed-loop system. How would you find the poles of the closed-loop sys?



The open-loop transfer function is: $G_{ol} = G_c G_p = \frac{(s+5)}{s(s^2+9)}$

The poles of the open loop transfer function are the roots of the open-loop characteristic polynomial: $s(s^2+9) = 0 \Rightarrow s = 0, \pm 3j$

The closed-loop transfer function is: $G_{cl} = \frac{G_c G_p}{1+G_c G_p}$



The characteristic polynomial of the closed-loop system: $1 + G_c G_p = 0$

The poles of the closed-loop system are the roots of the closed-loop characteristic polynomial:

$$1 + \frac{(s+5)}{s(s^2+9)} = s^3 + 10s + 5 = 0$$

In order to find the poles of the closed loop system we need to factor a 3rd degree polynomial.

This becomes hard, but fortunately, we can use the root-locus technique to treat such case.

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- What if we want to:

1. Observe the effect of changing gain parameters on the system response
   - *Where would the poles of the closed-loop system be as we change the gain* K
2. Observe the effect of adding dynamic compensation to the closed loop system
   - *What happens when we use a controller that adds poles and zeros to the closed-loop system?*

   Example: PI Controller, $G_c = K_P \frac{s+K_I/K_P}{s}$, *adds a zero and a pole*
3. Examine the sensitivity and stability of a closed loop system
   - *How close are the closed-loop poles to stability?*
4. Design controllers for higher order systems
   - *It is hard to factor roots for polynomials of 3rd and higher order.*

- A graphical controller design **technique**, such as the root-locus, can help us

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- The Laplace Function $F(s)$, is a function of the complex variable s, but how do we evaluate the function at any s?
- If we have the function in factored form

$$F(s) = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = \frac{(s+z_1)(s+z_2)\ \dots\ (s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)}$$

Where $\prod$ denotes product, m the number of zeros, n the number of poles

- The solution: $F(s) = M\angle\theta$

$$M = \frac{\prod(\text{zero vector lengths})}{\prod(\text{pole vector lengths})}$$

$$\theta = \sum \text{zero angles} - \sum \text{pole angles} = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+z_j)$$

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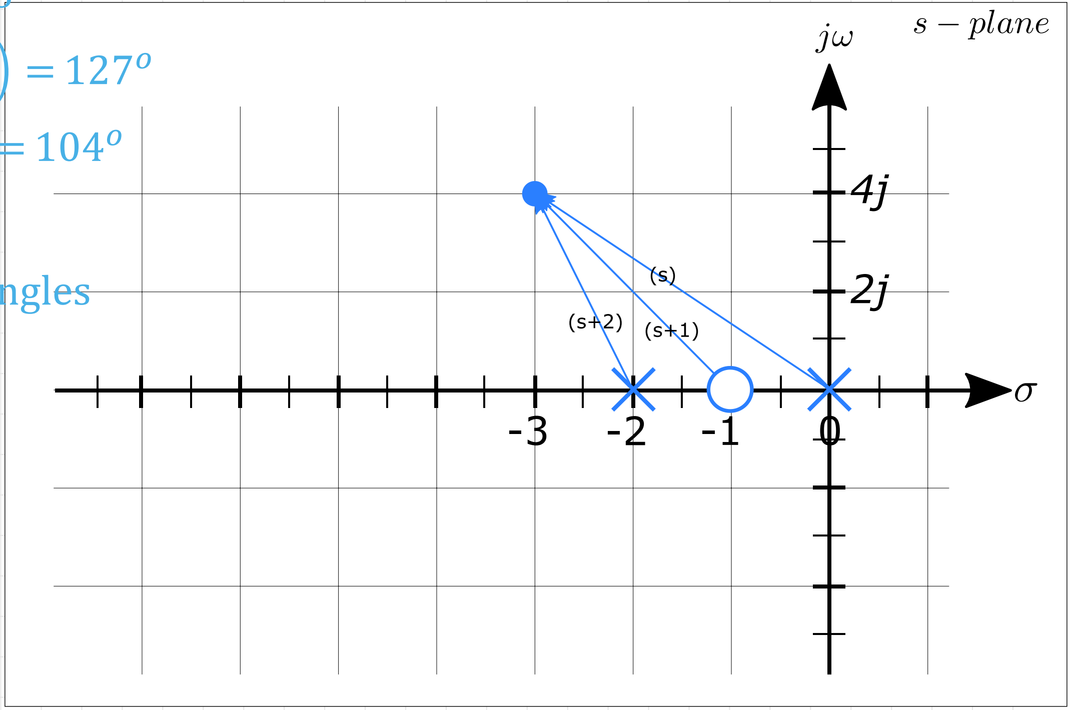
Evaluate, vectorially, the function $F(s) = \frac{(s+1)}{s(s+2)}$ at $s = -3 + j4$

To solve for this vectorially, we find the lengths and angles of all poles and zeros then add them

Zero: Length $= \sqrt{(-3-(-1))^2+(4-0)^2} = \sqrt{20}$, angle $= \tan^{-1}\left(\frac{4-0}{-3+1}\right) = 116^o$

Pole at 0: Length $= \sqrt{(-3-(0))^2+(4-0)^2} = \sqrt{25} = 5$, angle $= \tan^{-1}\left(\frac{4-0}{-3-0}\right) = 127^o$

Pole at -2: Length $= \sqrt{(-3-(-2))^2+(4-0)^2} = \sqrt{17}$, angle $= \tan^{-1}\left(\frac{4-0}{-3+2}\right) = 104^o$



$$F(s=-3+j4) = M\angle\theta = \frac{\prod(\text{zero vector lengths})}{\prod(\text{pole vector lengths})}\angle\sum \text{zero angles} - \sum \text{pole angles}$$

$$F(s=-3+j4) = \left(\frac{\sqrt{20}}{5\sqrt{17}}\right)\angle 116^o - 127^o - 104^o = -114^o$$

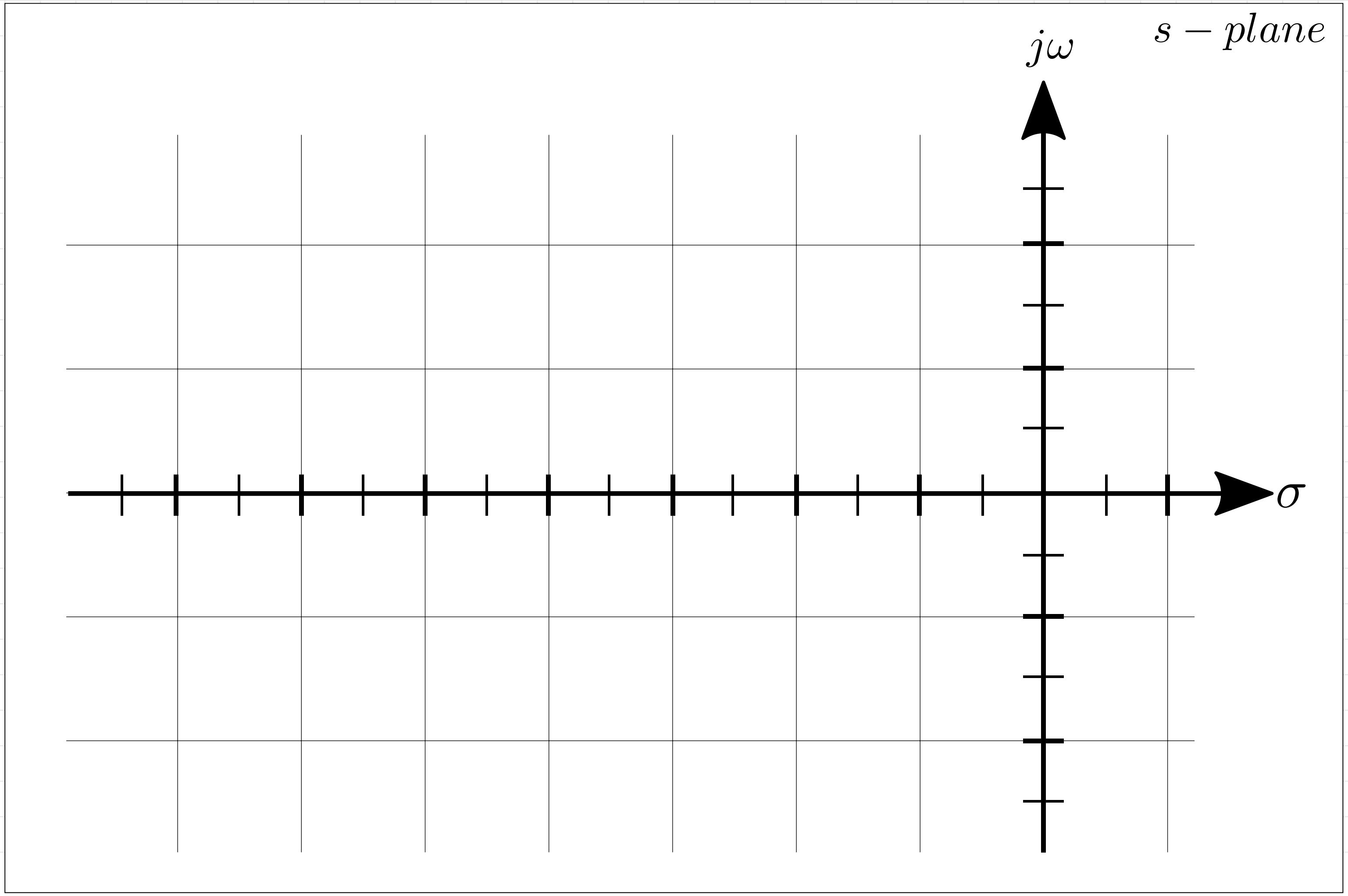
$$F(s=-3+j4) = 0.217\angle -114^o$$

Verify in MATLAB using *evalfr()*

$F = -0.089 - .197j$

Evaluate, vectorially, the function $F(s) = \frac{(s+3)(s+1)}{s(s+2)}$ at $s = -2 + j$

Algebraically: $F = 0.4 - 0.8j$



- Given the closed-loop transfer function
  - $G_{cl} = \frac{KG(s)}{1+KG(s)H(s)}$: non-unity feedback
- A pole, s, exists when the characteristic polynomial becomes zero:

$$1 + KG(s)H(s) = 0,$$

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^o, \quad k = 0, \pm1, \pm2, \pm3, \dots$$

Where -1 is represented in polar form as: $1\angle(2k+1)180^o$

- Also, the magnitude: $|KG(s)H(s)| = 1$
  - And if we assume $K \geq 0$ strictly. Then $K = \frac{1}{|G(s)||H(s)|}$ (eq. 1)
- And angle: $\angle KG(s)H(s) = (2k+1)180^o$ (eq. 2)

- In other words: for s to be a pole of the CL system it has to satisfy

  eq. 1 and eq. 2

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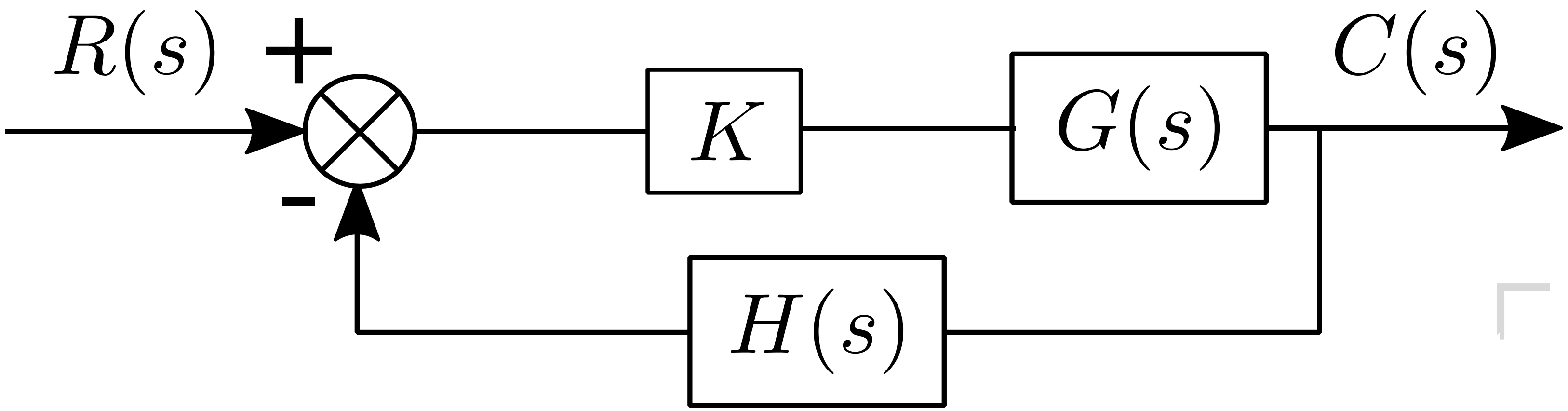
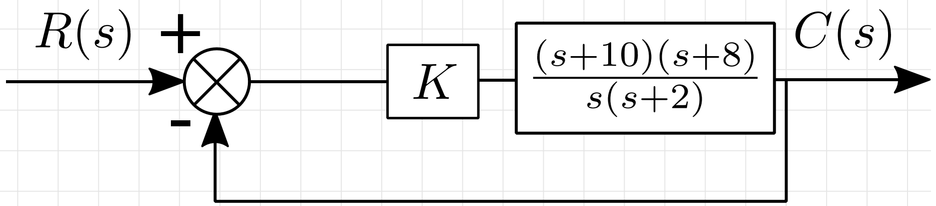
- The root locus is the locus of pole locations in the s-plane for varying values of gain $K \geq 0$, in the closed loop transfer function $G_{cl} = \frac{KG(s)}{1+KG(s)H(s)}$, that satisfy the following two conditions:
    1. Magnitude Condition: $K = \frac{1}{|G(s)||H(s)|}$
    2. Angle Condition: $\angle KG(s)H(s) = (2k+1)180^o$



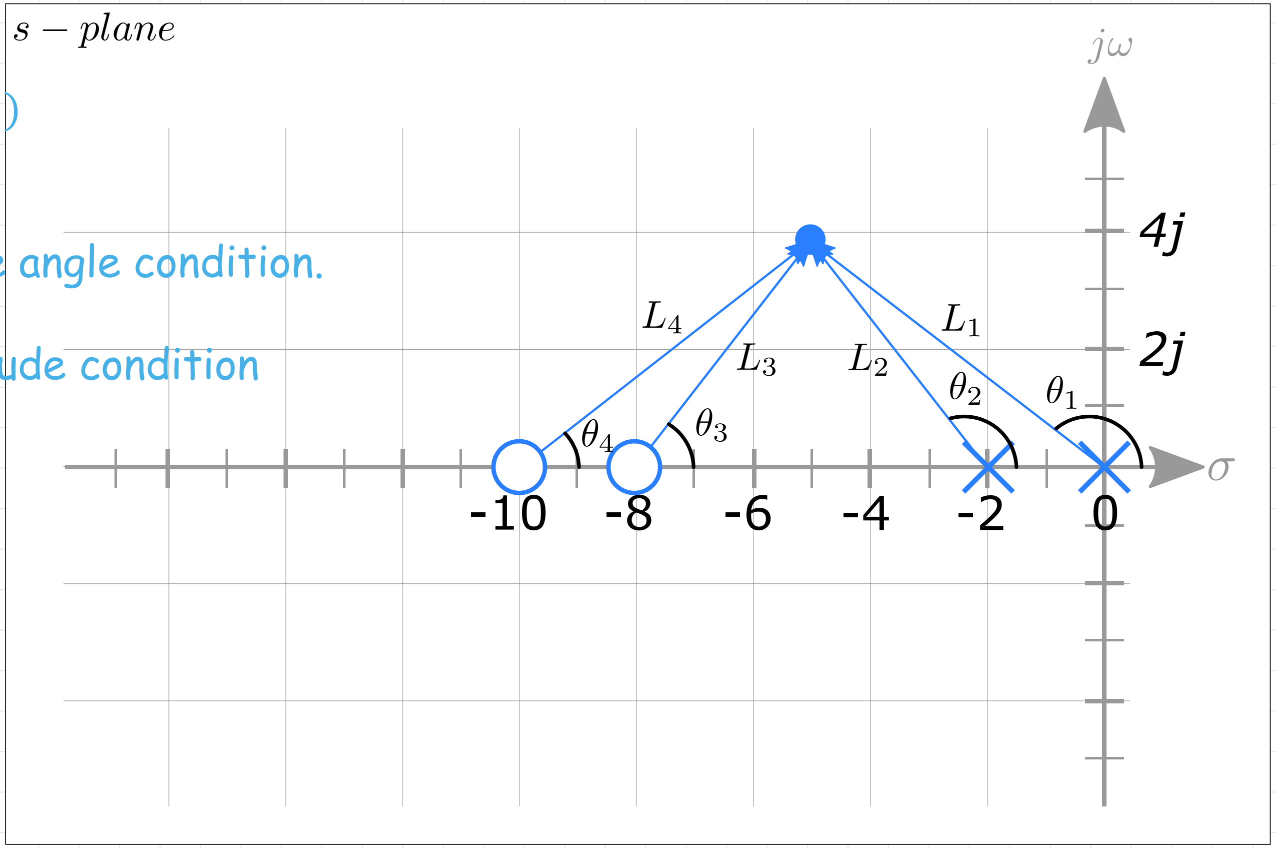
For the feedback system shown, show that the point $s = -5 + 3.87j$, is on the root-locus and find the value of the gain K at this point



To find if a point lies on the root-locus, given a feedback system open-loop (forward) transfer function. We check to see if the Angle Condition is met:

Angle Condition: $\angle\sum \text{zero angles} - \angle\sum \text{pole angles} = \angle|KG(s)| = \angle(2k+1)180^o$

$\angle|KG(s)| = \theta_3 + \theta_4 - \theta_1 - \theta_2$
$= \text{atan}(3.87/3) + \text{atan}(3.87/5) - (180 - \text{atan}(3.87/5))$
$-(180 - \text{atan}(3.87/3)) \approx 180 - 180 - 180 \approx -180^o$



The point is thus on the root locus, as it meets the angle condition.

To find the value of the gain, we check the magnitude condition

$$K = \frac{1}{|G(s)|} = \frac{\text{Pole Lengths}}{\text{Zero Lengths}} = \frac{L_1 L_2}{L_3 L_4} = 1$$

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For the feedback system shown, show that the point $s = -15$, is on the root-locus and find the value of the gain K at this point