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College of Engineering and Petroleum

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ME417 CONTROL OF MECHANICAL SYSTEMS

Part II: Controller Design Using Root-Locus

Lecture 2: Sketching the Root-Locus

Spring 2021

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- Objectives:
  - *Introduce guidelines on sketching the Root-Locus*
  - *Discuss methods of refining the Root-Locus*
  - *Discuss the use of the Root-Locus technique for varying different parameters*
- Reading:
  - *Nise: 8.4-8.5, 8.8*
- Practice problems included

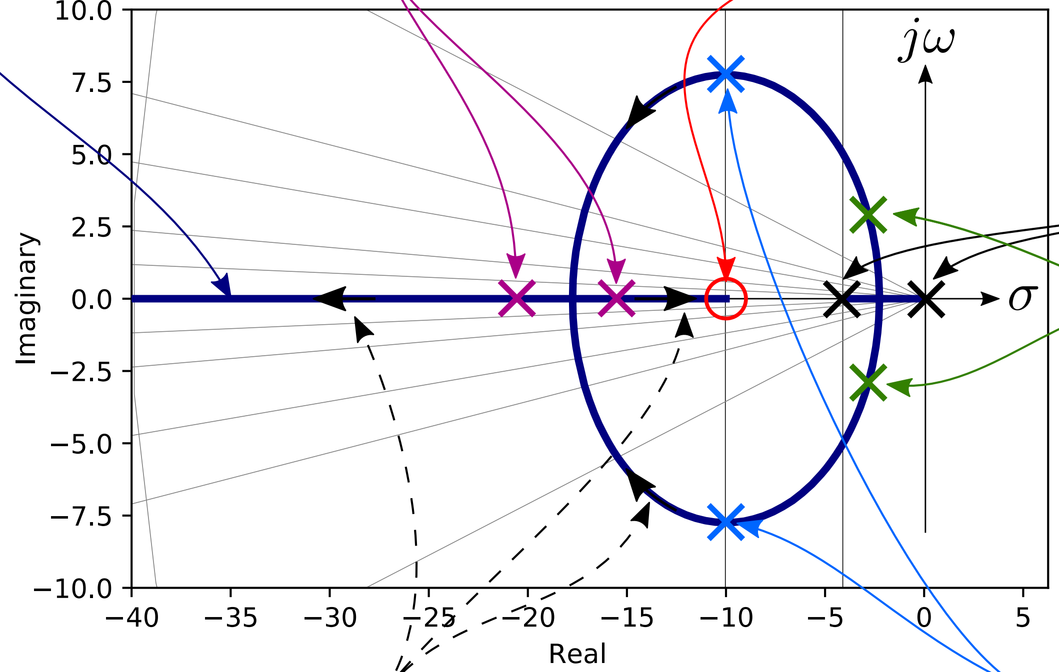
The Locus of the Closed-Loop Poles as K goes from 0 to infinity (Root-Locus)

Closed Loop Poles if K=K3

System Zero: CL Zeros = OL Zeros

Closed-Loop Pole if K= Infinity

Another System Zero at infinity



Open-Loop Poles:

Closed-Loop Poles if K=0

Closed Loop Poles if K=K1

Closed Loop Poles if K=K2

Closed-Loop Pole Location Motion as K increases:

As K increases, CL Poles move away from OL poles and toward system zeros:

Here: K1<K2<K3

$$G_{ol} = K\frac{(s+10)}{s(s+4)} \qquad G_{cl} = \frac{K(s+10)}{(s^2+(K+4)s+10K)}$$

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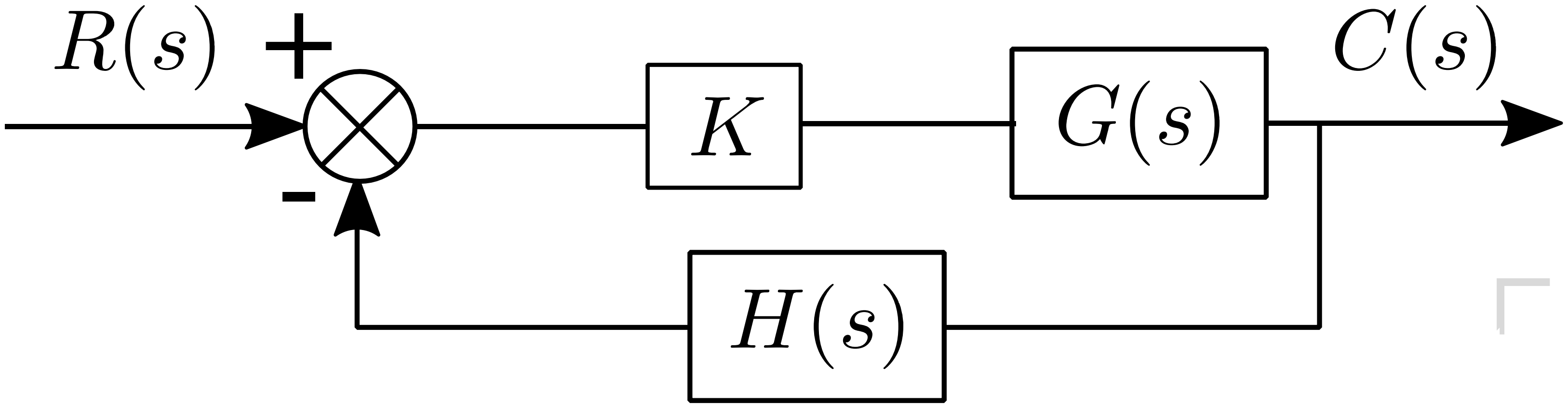
- The root locus is the locus of pole locations of the closed-loop transfer function $G_{cl} = \frac{KG(s)}{1+KG(s)H(s)}$,in the s-plane, for varying values of gain $K \geq 0$, that satisfy the following two conditions:
  1. Magnitude Condition: $K = \frac{1}{|G(s)||H(s)|}$
  2. Angle Condition: $\angle KG(s)H(s) = (2k+1)180^o$



- Given an open-loop transfer function, we can plot the Root-Locus by varying the value of gain K from $0 \rightarrow \infty$, calculating the values of the closed-loop poles and plotting them, forming the Root-Locus plot.
- This can be done numerically (e.g. *rlocus()* in MATLAB), but it becomes tedious to do it by hand.
- Instead, we can *sketch* the root locus by following a few basic sketching rules.

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Rules for Sketching the Root-Locus

- There are number of rules that, when followed, can help sketch the root-locus quite easily even for a high order transfer function.
- The first 5 rules can be used to rapidly sketch the root-locus by inspection, without any calculations; except for factoring the poles and zeros.
  - *You should be able to directly sketch an approximate root-locus using these rules, just by inspecting the open-loop transfer function.*
- The remaining rules are for **refining** the sketch and would require some calculations.

1. Number of Branches:
   - *Number of branches = Number of closed-loop poles*
2. Symmetry:
   - *The Root-Locus is symmetric about the real axis*
3. Real-Axis Segments:
   - *On the real axis, the Root-Locus exists to the left of an odd number of real-axis open-loop pole or real-axis finite open-loop zero*
4. Starting and Ending Points:
   - *The Root-Locus begins at poles* ($K = 0$) *and ends at zeros* ($K = \infty$)
5. Behavior at Infinity:
   - *The Root-Locus approaches straight line asymptotes as the Root-Locus approaches infinity*

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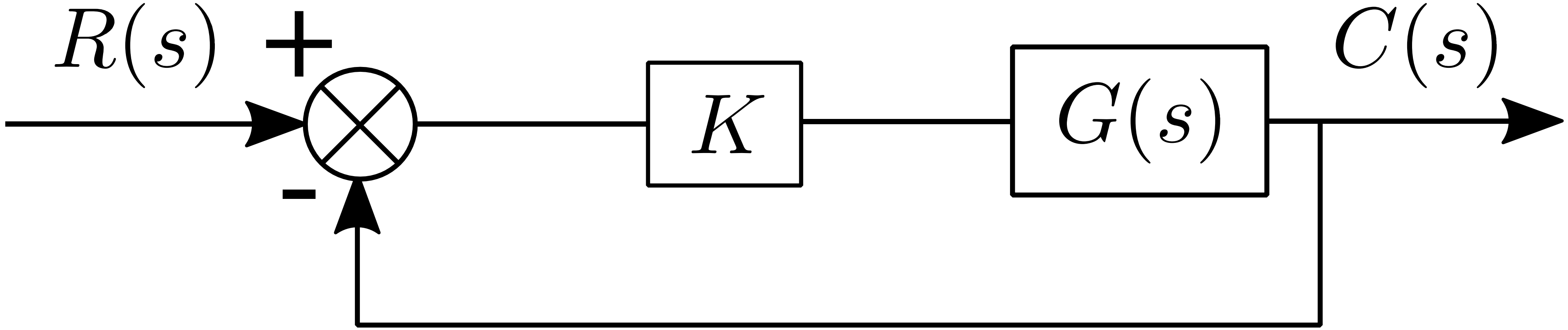
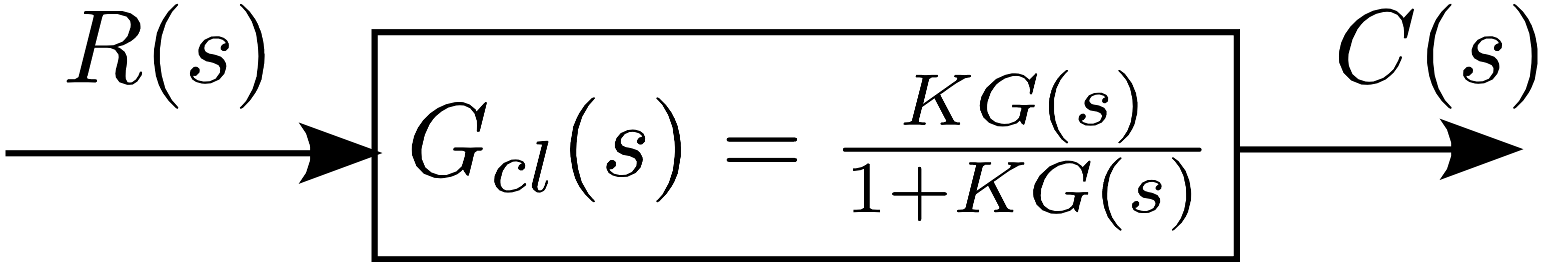
- Let’s review the Root-Locus sketching rules for a unity feedback system
  - The open-loop (forward) transfer function of the feedback system is:

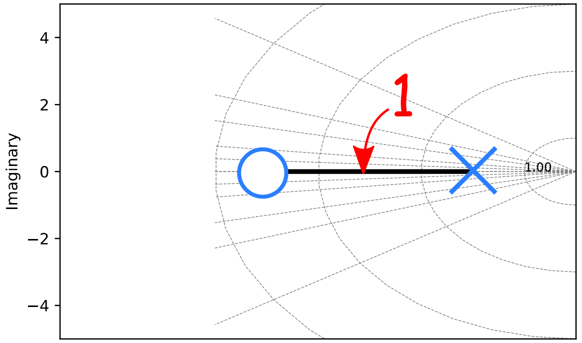
$$G_{OL} = KG(s)$$

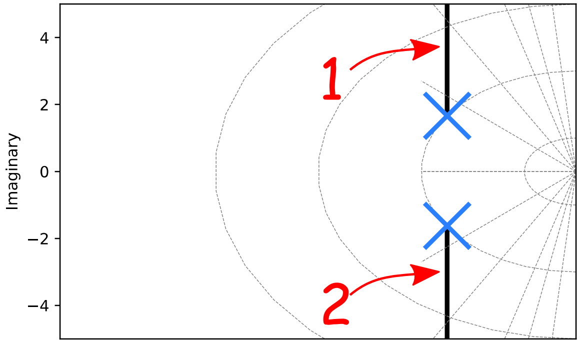


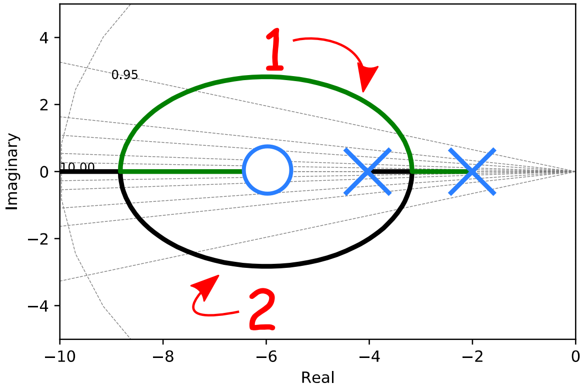


- *Number of branches of the Root-Locus equals the number of closed-loop poles*

Open-Loop Transfer Function	Closed-Loop Transfer Function
$\dfrac{K(s+6)}{s+2}$	$\dfrac{K(s+6)}{K(s+6)+s+2}$
$\dfrac{K}{s^2+5s+9}$	$\dfrac{K}{K+s^2+5s+9}$
$\dfrac{K(s+6)}{(s+2)(s+4)}$	$\dfrac{K(s+6)}{K(s+6)+(s+2)(s+4)}$







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- *The Root-Locus is symmetric about the real axis*

Open-Loop
Transfer Function

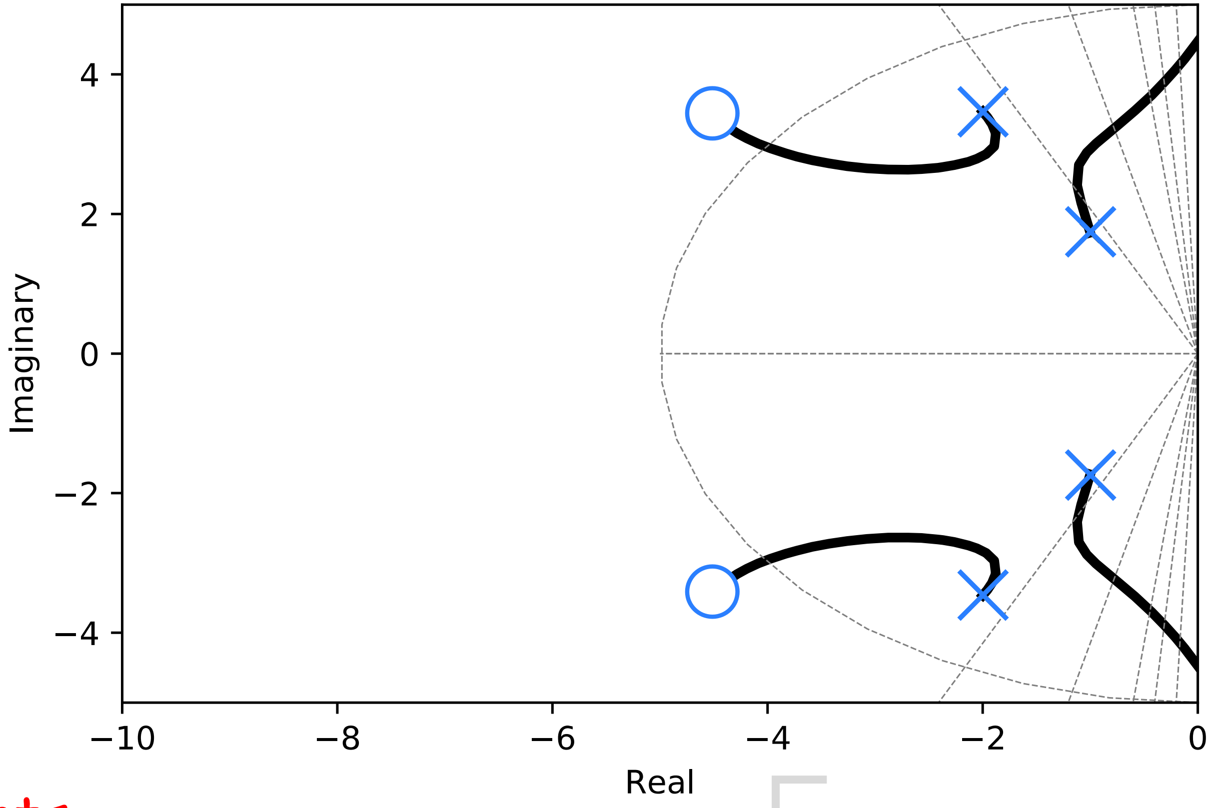
$$\frac{K\left(s^2+9s+32\right)}{(s^2+2s+4)(s^2+4s+16)}$$

4 Complex Poles

Closed-Loop
Transfer Function

$$\frac{K\left(s^2+9s+32\right)}{K(s^2+9s+32)+(s^2+2s+4)(s^2+4s+16)}$$

4th Order CL System: 4 Segments



- *On the real axis, the Root-Locus exists to the left of an odd number of real-axis open-loop pole or real-axis finite open-loop zero*

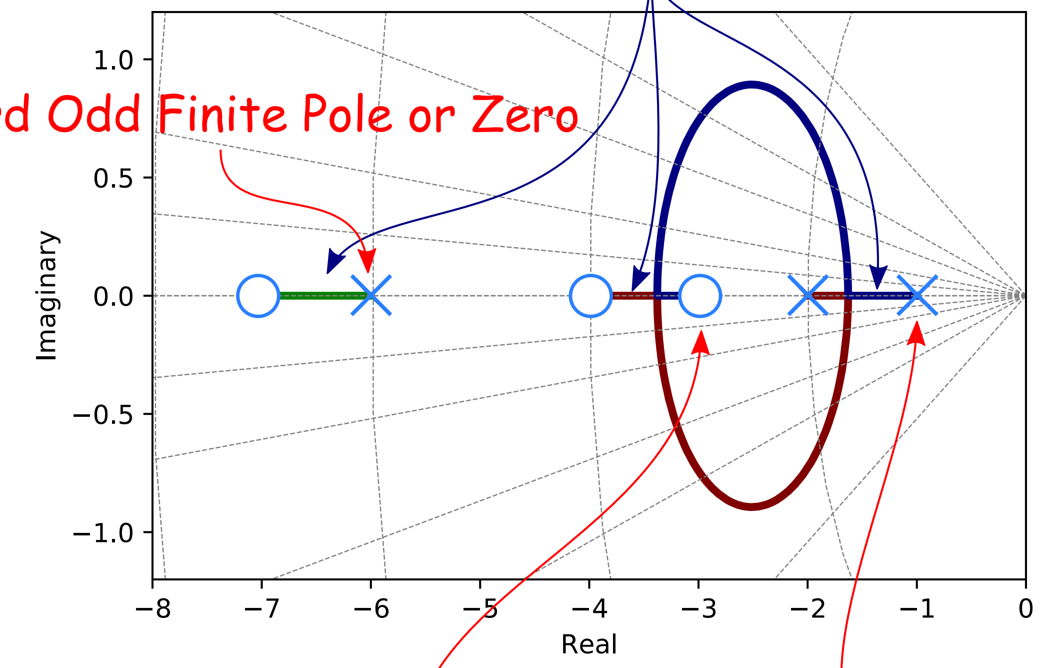
Open-Loop
Transfer Function

$$\frac{K(s+3)(s+4)(s+7)}{(s+1)(s+2)(s+6)}$$

Closed-Loop
Transfer Function

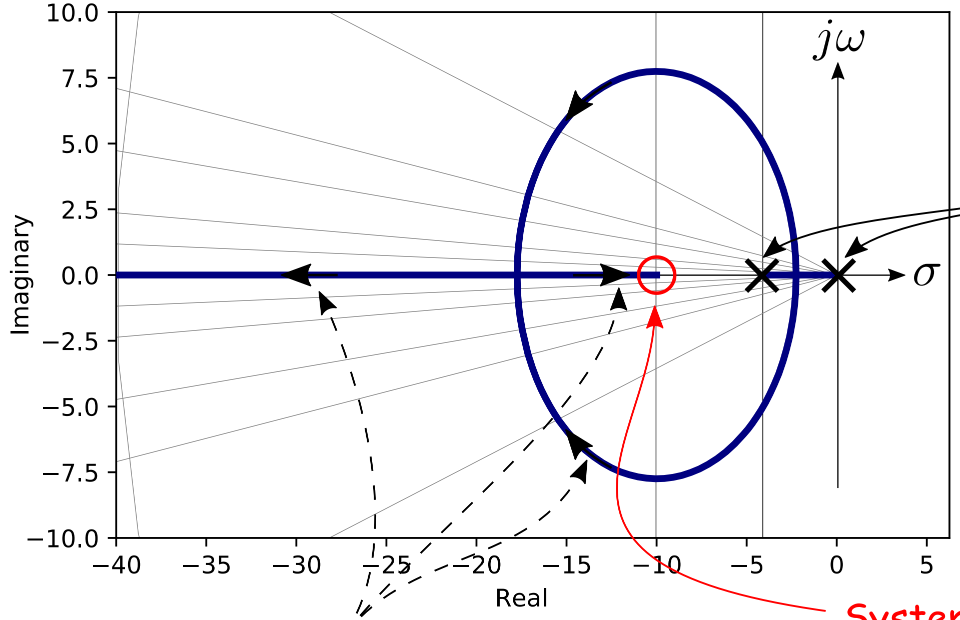
$$\frac{K(s+3)(s+4)(s+7)}{K(s+3)(s+4)(s+7)+(s+1)(s+2)(s+6)}$$

Root-Locus segments, on the real axis, exist only to the left of odd finite poles or zeros



- *The Root-Locus begins at the finite and infinite poles of the open-loop transfer function (where $K = 0$), and ends at the finite and infinite zeros of the open-loop transfer function (where $K = \infty$)*
- If there are n poles and m zeros, where $n > m$. There are $n - m$ infinite zeros

Another System Zero at infinity



Open-Loop Poles:
Closed-Loop Poles if K=0

System Zero: CL Zeros = OL Zeros
Closed-Loop Pole if K= Infinity

Closed-Loop Pole Location Motion as K increases:
As K increases, CL Poles move away from OL poles and toward system zeros:
Here: K1<K2<K3

$$G_{ol} = K\frac{(s+10)}{s(s+4)} \qquad G_{cl} = \frac{K(s+10)}{(s^2+(K+4)s+10K)}$$

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Root-Locus Sketching Rule #5: Behavior at Infinity

- *The Root-Locus approaches straight lines asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept,* σ_a *and angle,* θ_a *as follows:*

$$\sigma_a = \frac{\sum Finite\ Poles - \sum Finite\ Zeros}{\#Finite\ \ Poles\ - \#Finite\ Zeros}$$

$$\theta_a = \frac{(2k+1)\pi}{\#Finite\ Poles\ - \#Finite\ Zeros}$$

Where $k = 0, \pm 1, \pm 2, \ldots$ *and the angle is given in radians with respect to the positive extension of the real axis.*

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- Consider the feedback system shown.
- Three zeros at infinity: Three asymptotes
- Real Axis Intercept:

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

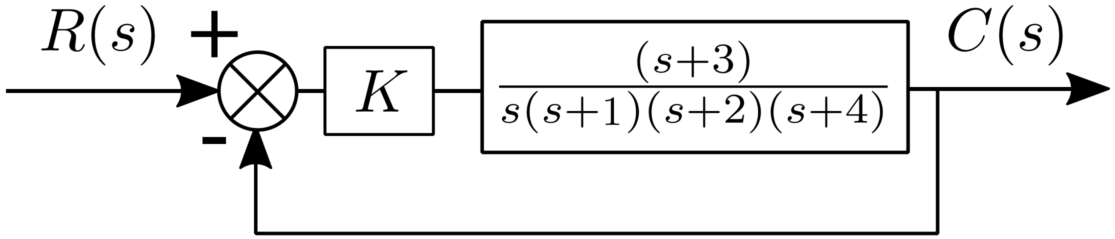
- Slopes' angles:

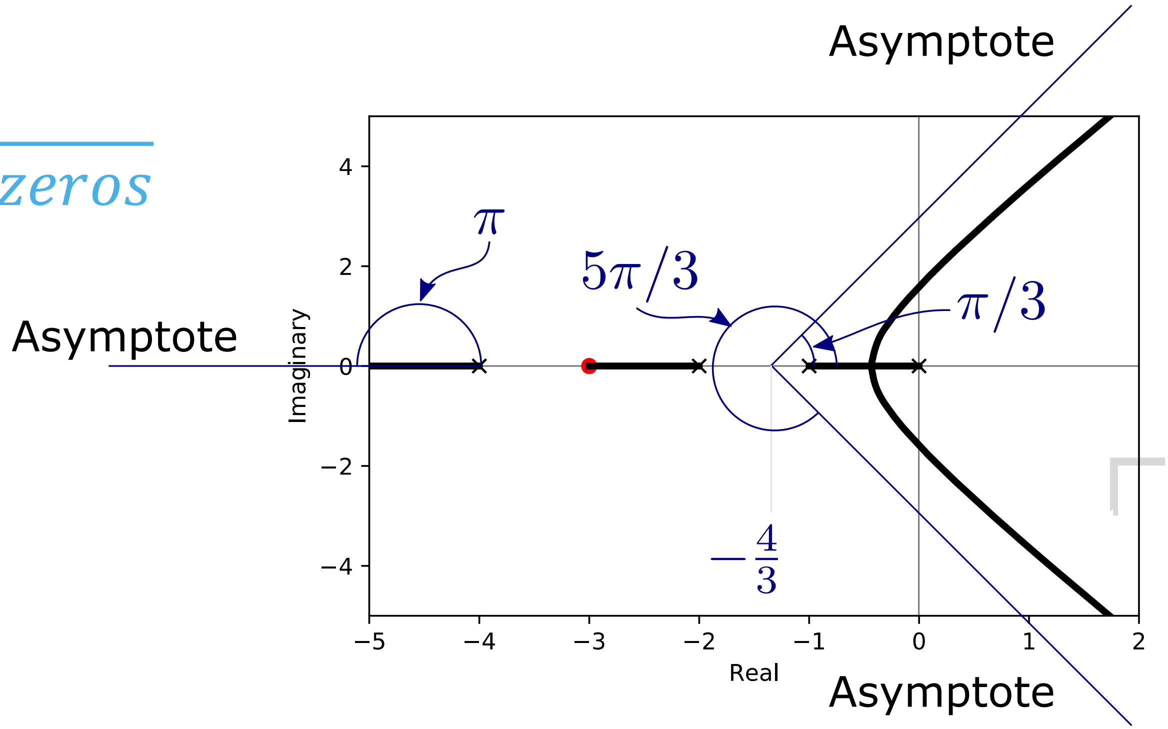
$$\theta_a = \frac{(2k+1)\pi}{\#finite\ poles\ - \#finite\ zeros}$$

$$\theta_a = \frac{\pi}{4-1} = \frac{\pi}{3}, k=0$$

$$\theta_a = \frac{3\pi}{4-1} = \pi, k=1$$

$$\theta_a = \frac{5\pi}{4-1} = \frac{5\pi}{3}, k=2$$





Root-Locus Sketching Rule #5: Behavior at Infinity

- Note that the asymptote angles can be obtained by quick inspection
- If n the number of poles and m the number of zeros of the open-loop transfer function:
- If $n = m$: No Asymptotes
- If $n - m = 1$: 1 zero at ∞, 1 asymptote with $\theta_a = \pi$
- If $n - m = 2$: 2 zeros at ∞, 2 asymptotes with $\theta_a = \frac{\pi}{2}, \theta_a = \frac{3\pi}{2}$
- If $n - m = 3$: 3 zeros at ∞, 3 asymptotes with $\theta_a = \frac{\pi}{3}, \theta_a = \pi, \theta_a = \frac{5\pi}{3}$
- If $n - m = 4$: 4 zeros at ∞, 4 asymptotes with $\theta_a = \frac{\pi}{4}, \theta_a = \frac{3\pi}{4}, \theta_a = \frac{5\pi}{4}, \theta_a = \frac{7\pi}{4}$

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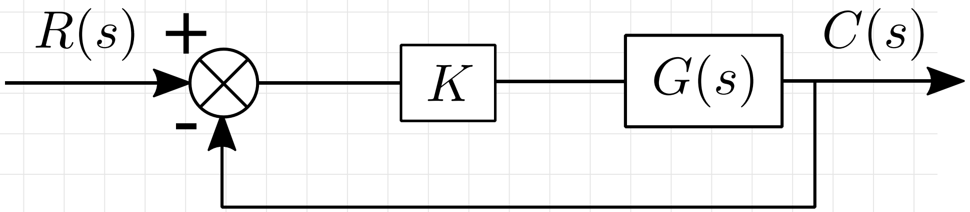
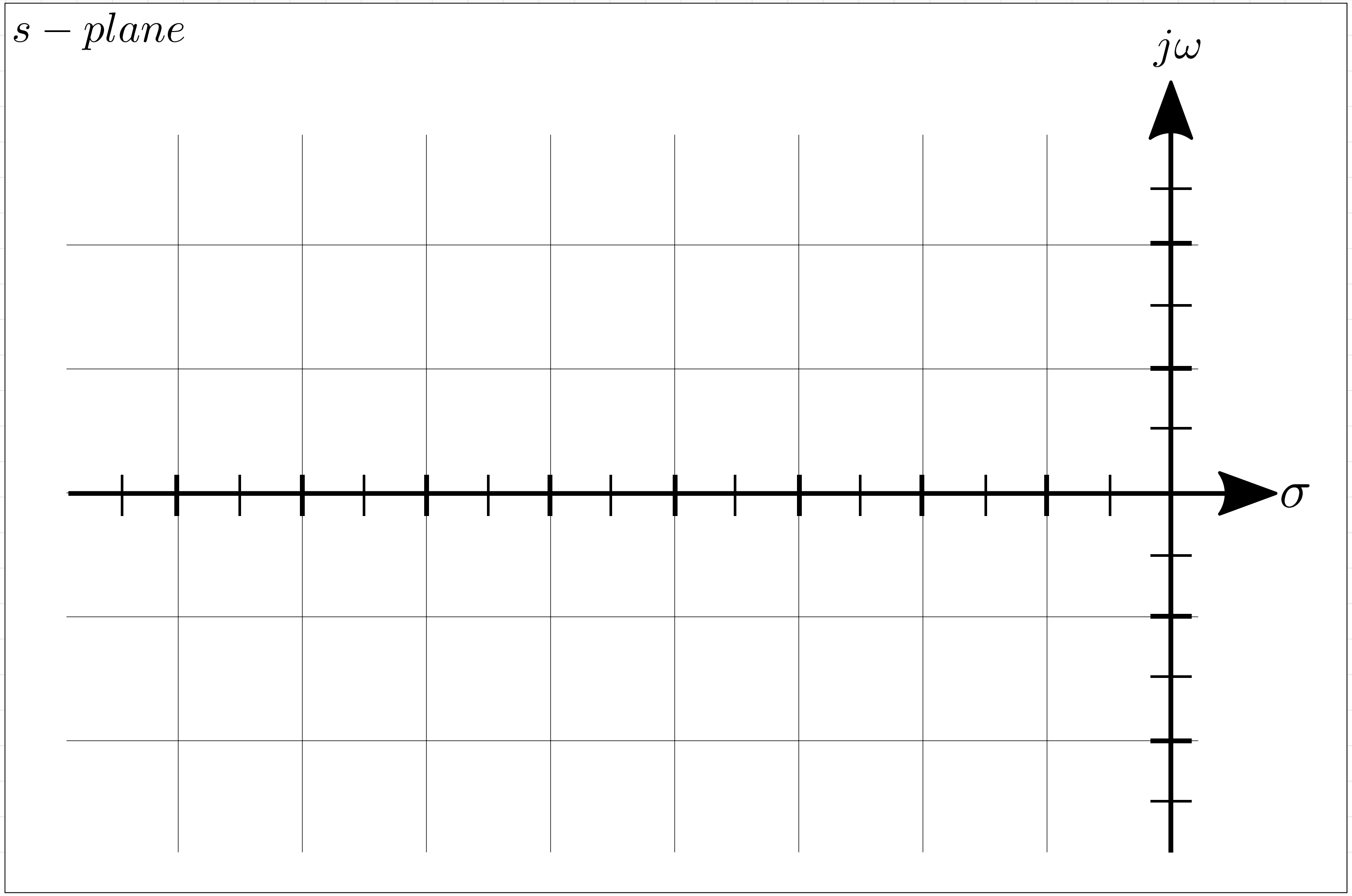
Example 1

Sketch the root locus, by inspection, for the following system in a unity feedback loop.

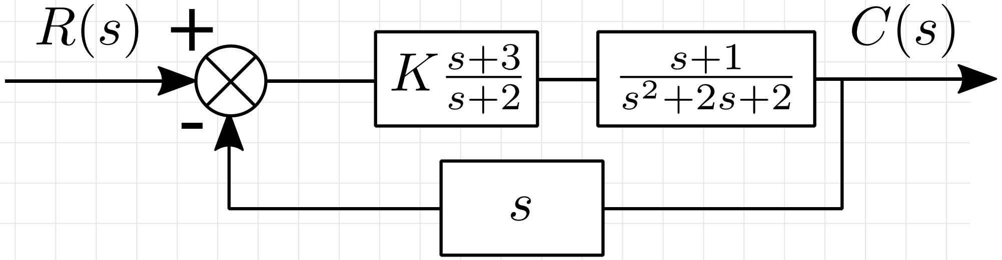


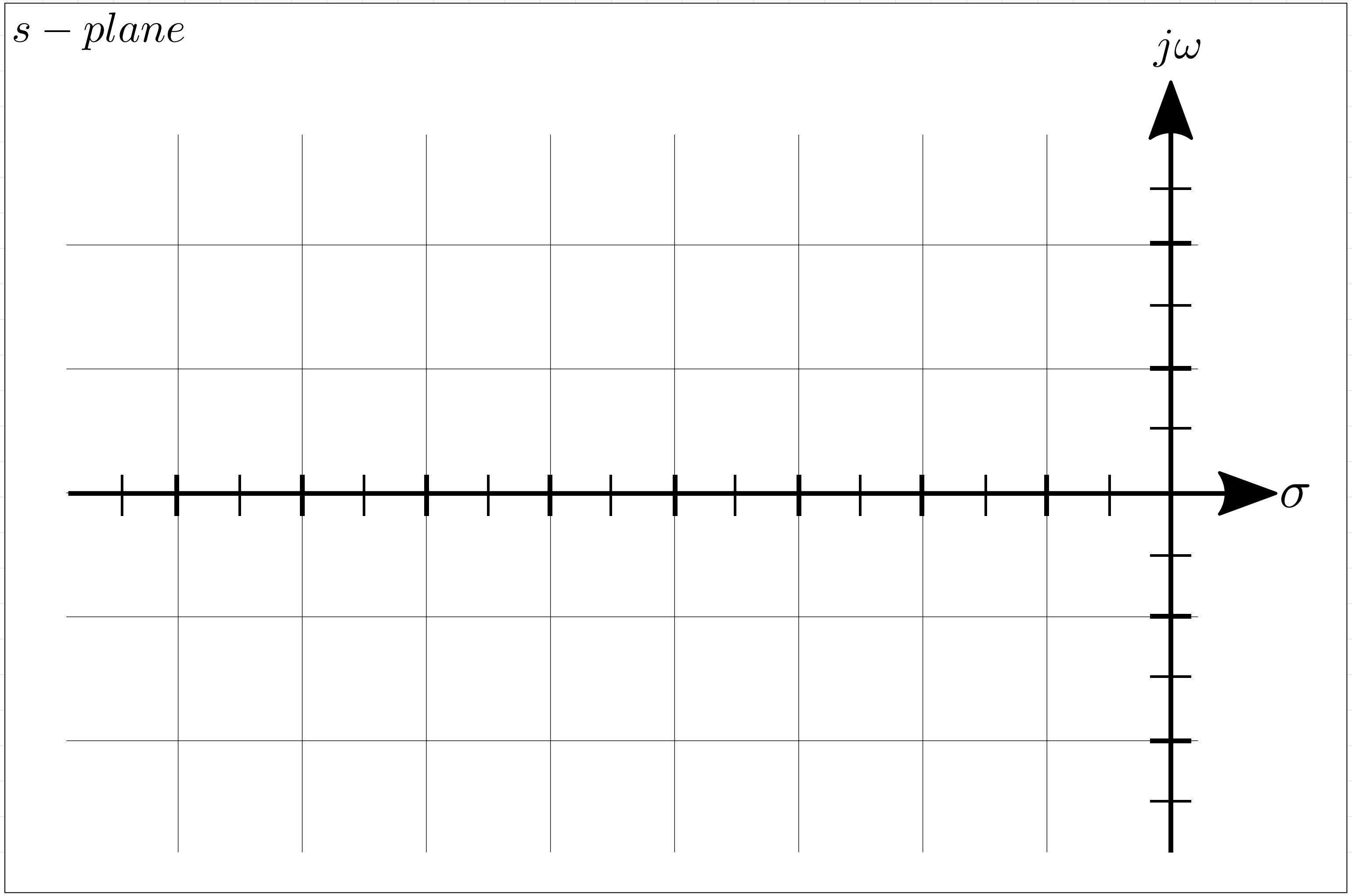
$$G(s) = \frac{(s+7)}{(s+6)(s+1)}$$



Example 2

Sketch the root locus, by inspection, for the following non-unity feedback system.

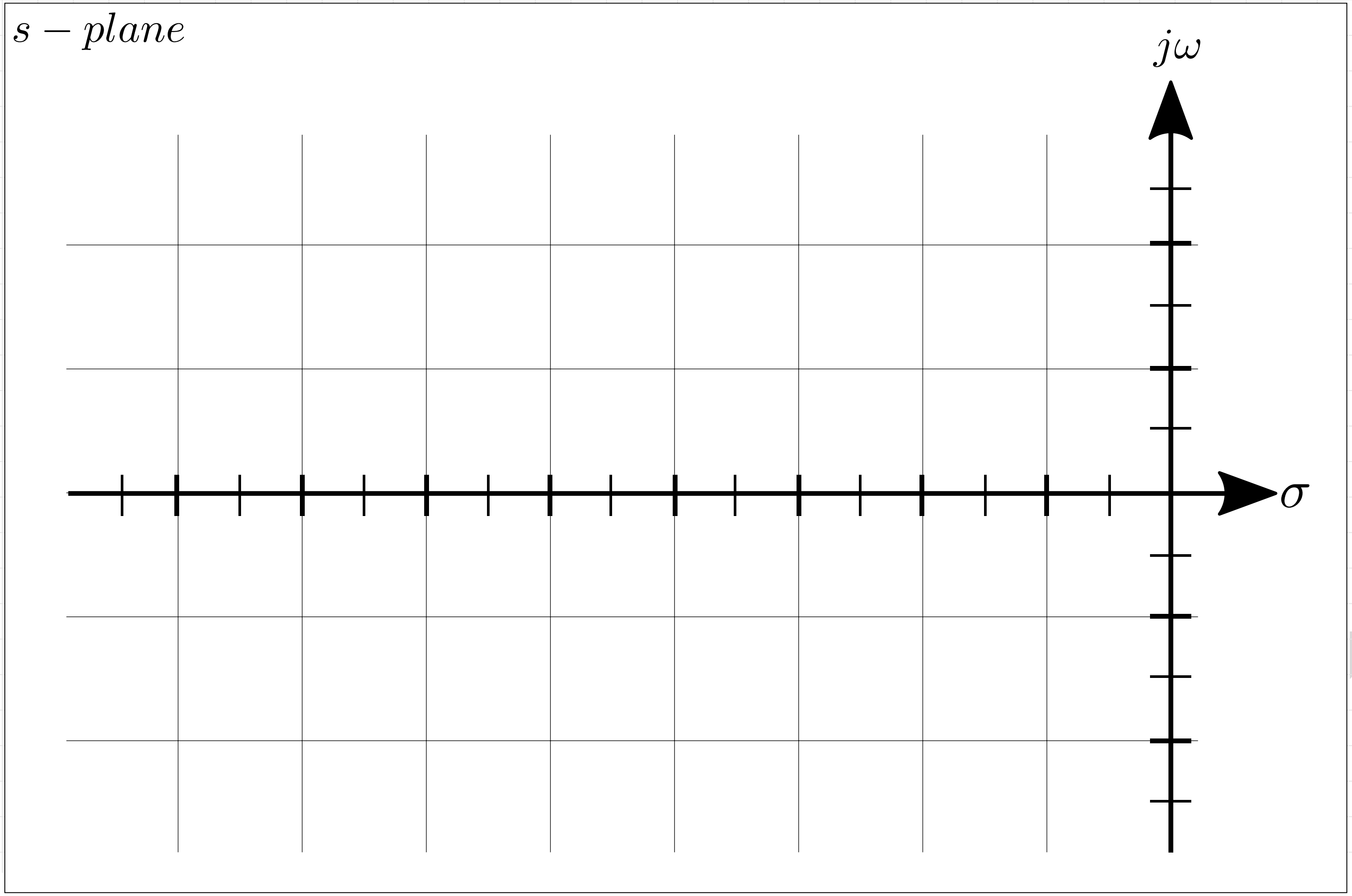




Example 3

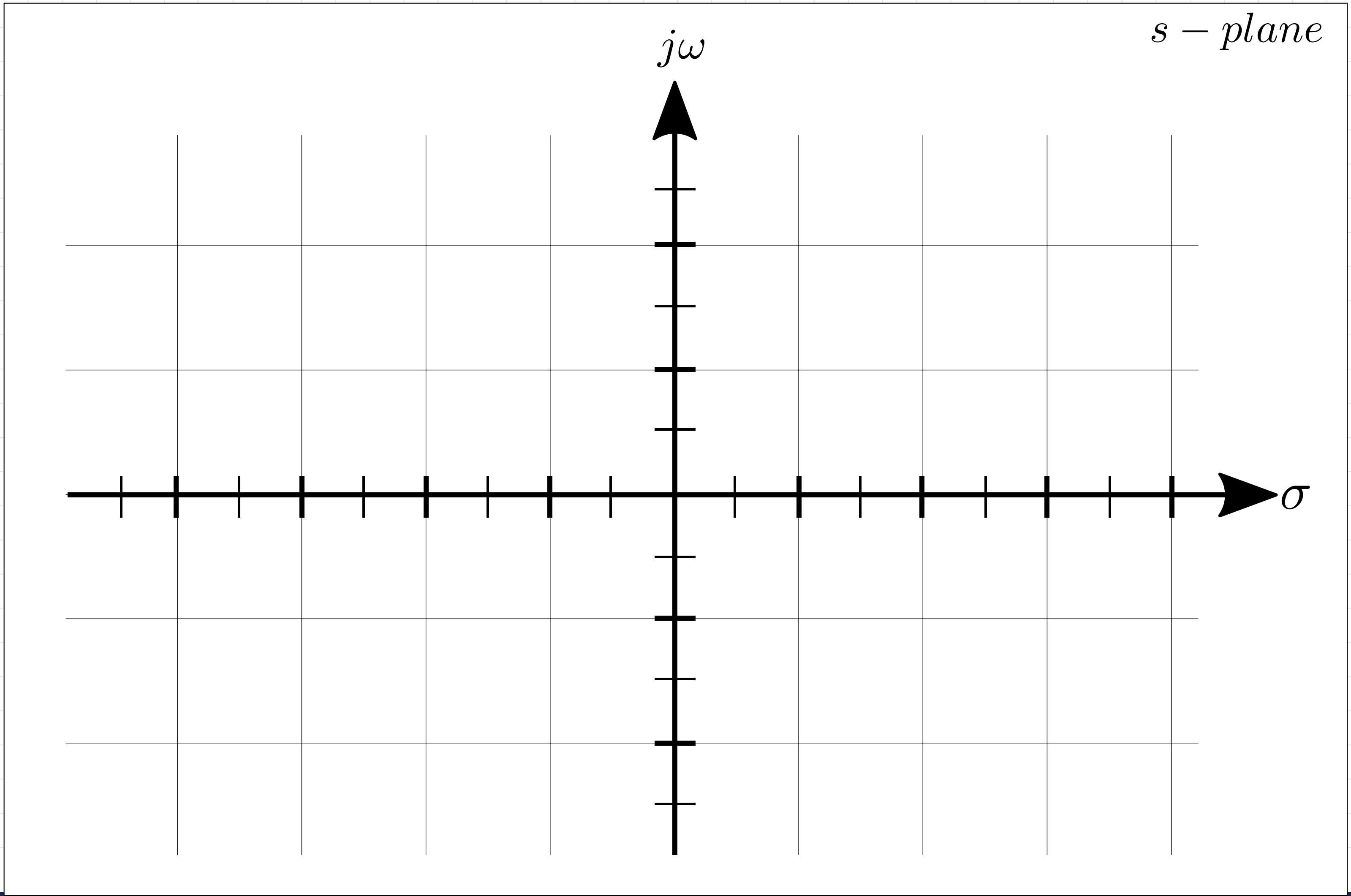
Sketch the root locus, by inspection, for the following open-loop transfer function, in a unity feedback system.

$$G(s) = K\frac{(s+6)}{(s+1)(s+2)(s+3)}$$



Sketch the root locus, by inspection, for the following open-loop transfer function, in a unity feedback system.

$$G(s) = K\frac{(s-2)(s-5)}{(s+1)(s+2)(s+3)}$$



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With practice, the first 5 rules should be applied by inspection, the following rules can be used to refine the root-locus sketch

6. Real-axis break-away and break-In points
   - *The root-locus breaks away from the real-axis at point of max gain and breaks in at point of min gain.*
7. Calculation of the $j\omega$ axis crossing
   - *The RL crosses the* $j\omega$ *axis when* $G(s) = G(j\omega), s = 0 + j\omega$
8. Angles of departure and arrival
   - *The RL departs from complex open-loop poles and arrives at complex open-loop zeros at angles that can be calculated.*
9. Plotting and calibrating the root locus
   - *All the points on the RL satisfy the relationship*

   $\angle G(s)H(s) = (2k+1)180^o$

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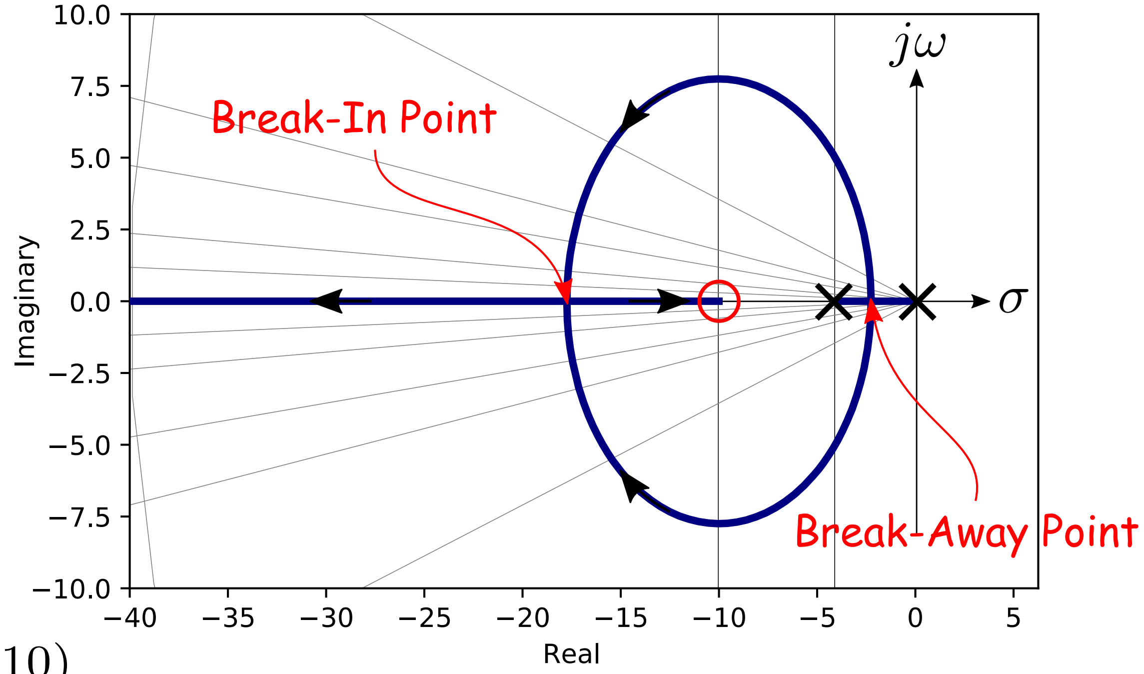
- Break-away points exists when there is a root-locus segment between two poles on the real-axis
- Break-in points exists when there is a root-locus segment between two zeros on the real axis.

Another System Zero at infinity



$$G_{ol} = K\frac{(s+10)}{s(s+4)}$$

$$G_{cl} = \frac{K(s+10)}{(s^2+(K+4)s+10K)}$$

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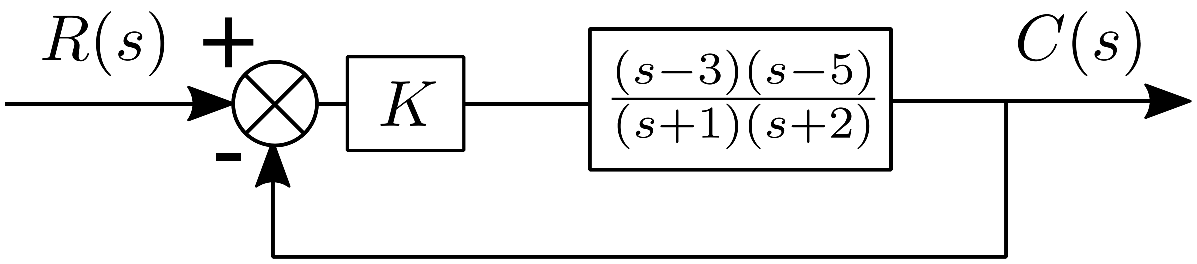
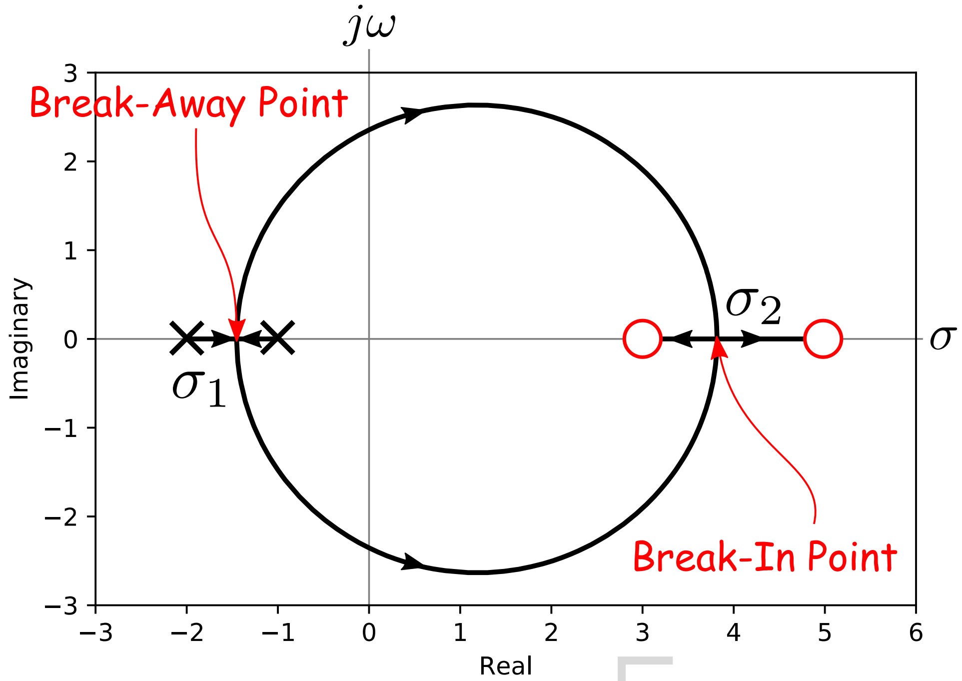
Root-Locus Sketching Rule #6: Real-Axis break-away and break-In points

- The break-away point occurs at the point with maximum gain on the real axis segment.
  - Remember that the CL poles move **away** from the OL poles with increasing K
- The break-in point occurs at the point with minimum gain on the real axis segment.
  - Remember that the CL poles move **toward** the OL zeros with increasing K
- To find the break-away and break-in points, we use the closed-loop characteristic polynomial and differentiate the gain with respect to $s = \sigma$, we get values of σ which correspond to the break-away and break-in points.

- Consider the feedback system shown



- CL char. poly. : $1+KG(s) = 1 + \frac{K(s-3)(s-5)}{(s+1)(s+2)} = 0$
- $\frac{K(s-3)(s-5)}{(s+1)(s+2)} = -1,\ K = \frac{-(s+1)(s+2)}{(s-3)(s-5)}$,
- Substitute $s = \sigma$ to express the gain on the real-axis only, since $\omega j = 0$:
- $K = \frac{-(\sigma+1)(\sigma+2)}{(\sigma-3)(\sigma-5)} = \frac{-(\sigma^2+3\sigma+2)}{(\sigma^2-8\sigma+15)} = -1$
- The above function for K should give two discontinuous curves
- Differentiate K w.r.t to σ to find min/max
- $\frac{dK}{d\sigma} = \frac{(11\sigma^2-26\sigma-61)}{(\sigma^2-8\sigma+15)^2} = 0$, gives $\sigma = -1.45, 3.82$
- Break-away point $\sigma_1 = -1.45$
- Break-in point $\sigma_2 = 3.82$



What happens to the closed-loop system here as we increase K?

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Root-Locus Sketching Rule #7: Calculation of the $j\omega$ axis crossing

- The imaginary axis crossing occurs when the real component $\sigma = 0$
- To find the value of gain K where the crossing occurs, we sub $s = j\omega$ in the characteristic polynomial and solve for K (Positive values of K only, since we treat negative feedback systems)
- Given the characteristic polynomial: $KG(s)H(s) = -1$
  - Solve for K in $KG(j\omega)H(j\omega) = -1$, to find the $j\omega$ crossing location
  - Finding both the value of the gain K and the $j\omega$ intercept value of ω

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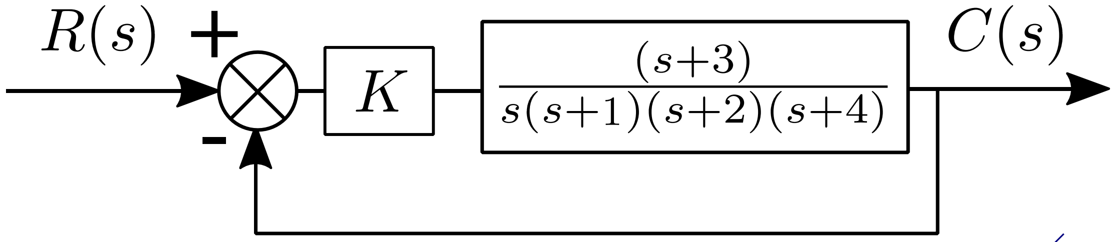
- Consider the feedback system shown.



- $KG(s)H(s) = \frac{K(s+3)}{s^4+7s^3+14s^2+8s}$
- Substitute $s = j\omega$, and simplify:

$$KG(j\omega)H(j\omega) = \frac{(jK\omega + 3K)}{\omega^4 - j7\omega^3 - 14\omega^2 + j8\omega} = -1$$

- Gives:

$$-\omega^4 + j7\omega^3 + 14\omega^2 - j(8+K)\omega - 3K = 0$$

- Separate the real from the fake (j/k: imaginary):

$$real: -\omega^4 + 14\omega^2 - 3K = 0$$

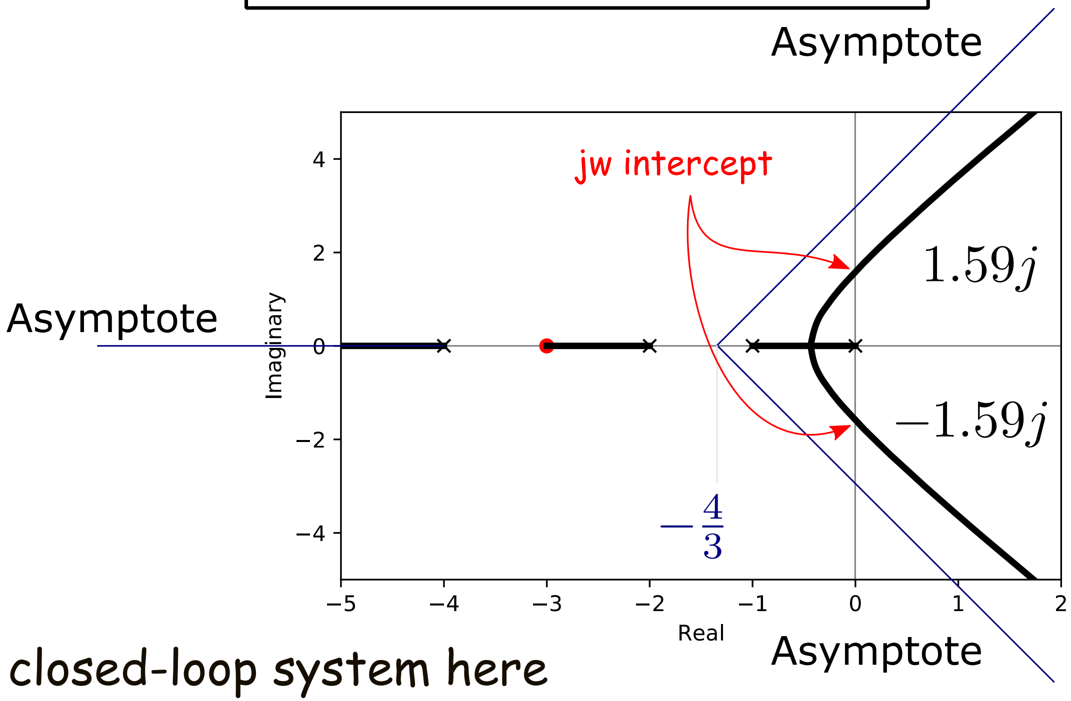
$$imag: 7\omega^3 - (8+K)\omega = 0$$

- From imag.: $\omega^2 = \frac{8+K}{7}$, subs in real:

$$-\left(\frac{8+K}{7}\right)^2 + 14\left(\frac{8+K}{7}\right) - 3K = 0$$

$$K^2 + 65K - 720 = 0 \Rightarrow K = -74.6, 9.65,$$

Take the positive $K = 9.65$, $\omega = \sqrt{\frac{8+9.65}{7}} = 1.59 rad/s$



What happens to the closed-loop system here when K places the CL poles at the intercept?

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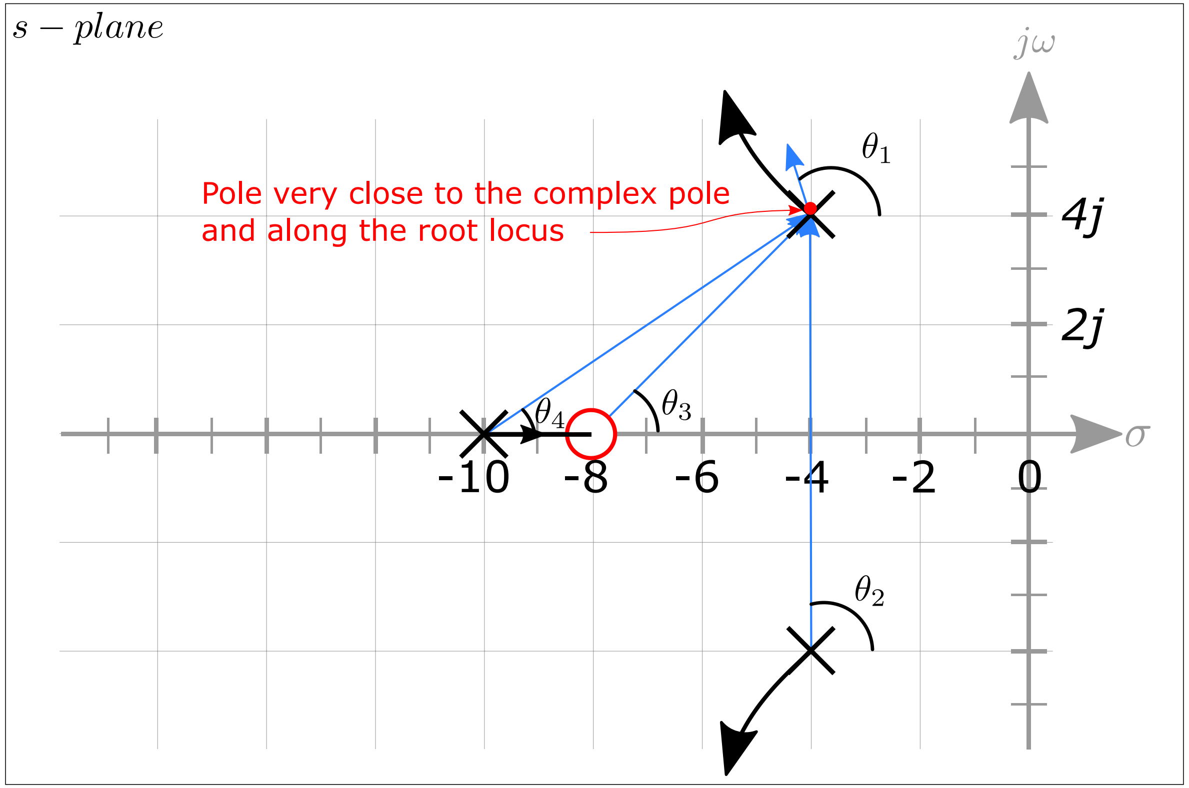
- To find the angle of departure of a complex pole, we choose a CL pole location very close to the complex pole, then satisfy the angle condition:
  - $\angle KG(s)H(s) = \pm(2k+1)180^o$

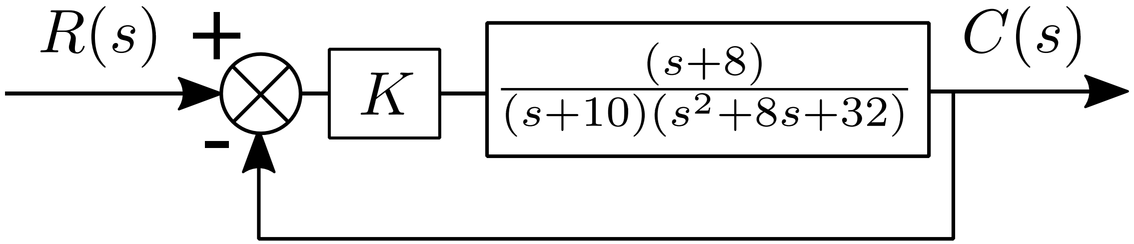
$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = -180^o$$

$$\theta_1 = 180^o - 90^o + \tan^{-1}\left(\frac{4}{4}\right) - \tan^{-1}\left(\frac{4}{6}\right)$$

$$\theta_1 = 180^o - 90^o + 45^o - 33.69^0 = 101.31^o$$

- Thus the angle of departure of the root-locus from the pole at $s = -4 + 4j$ is $\theta = 101.31^o$





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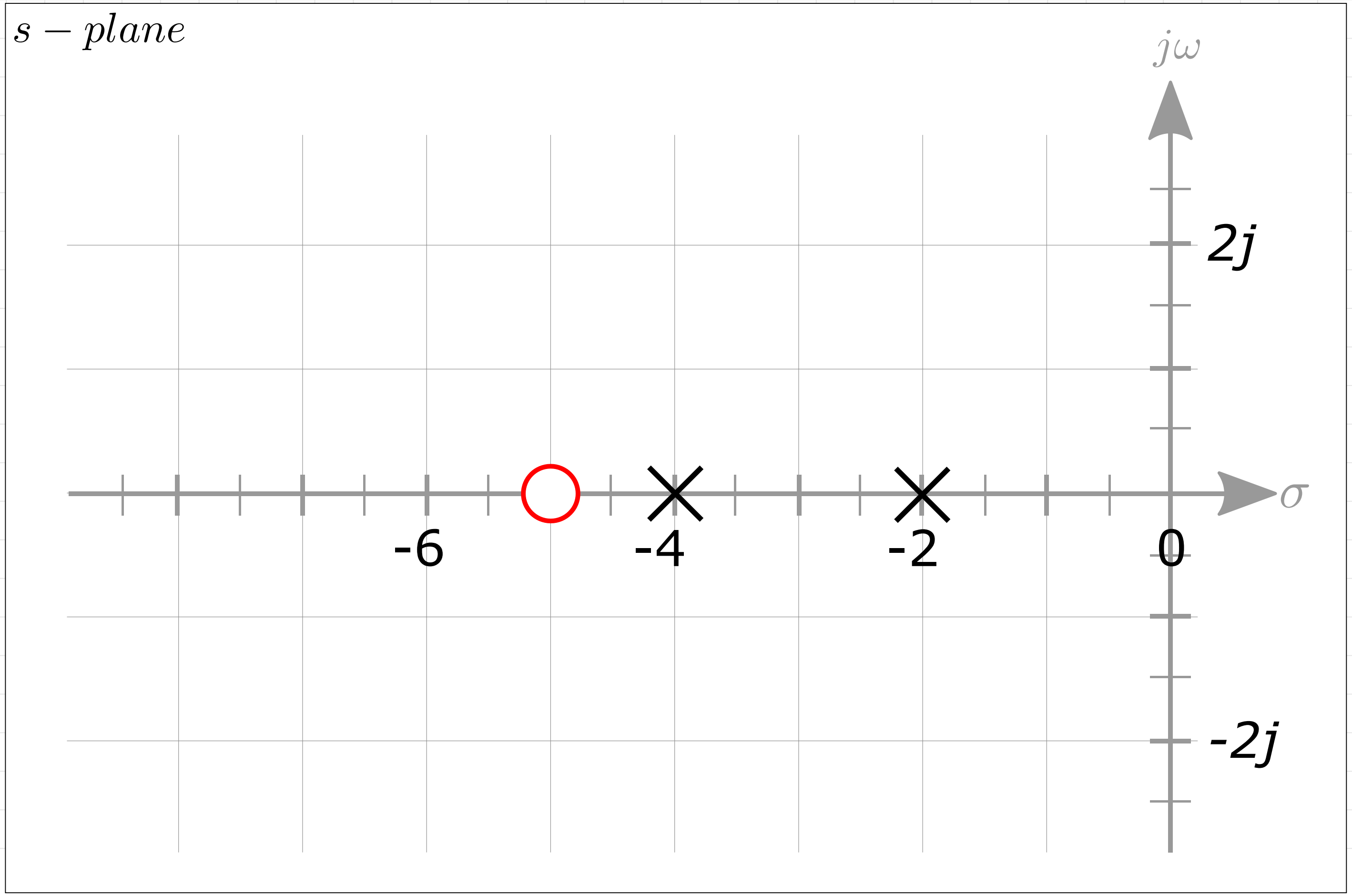
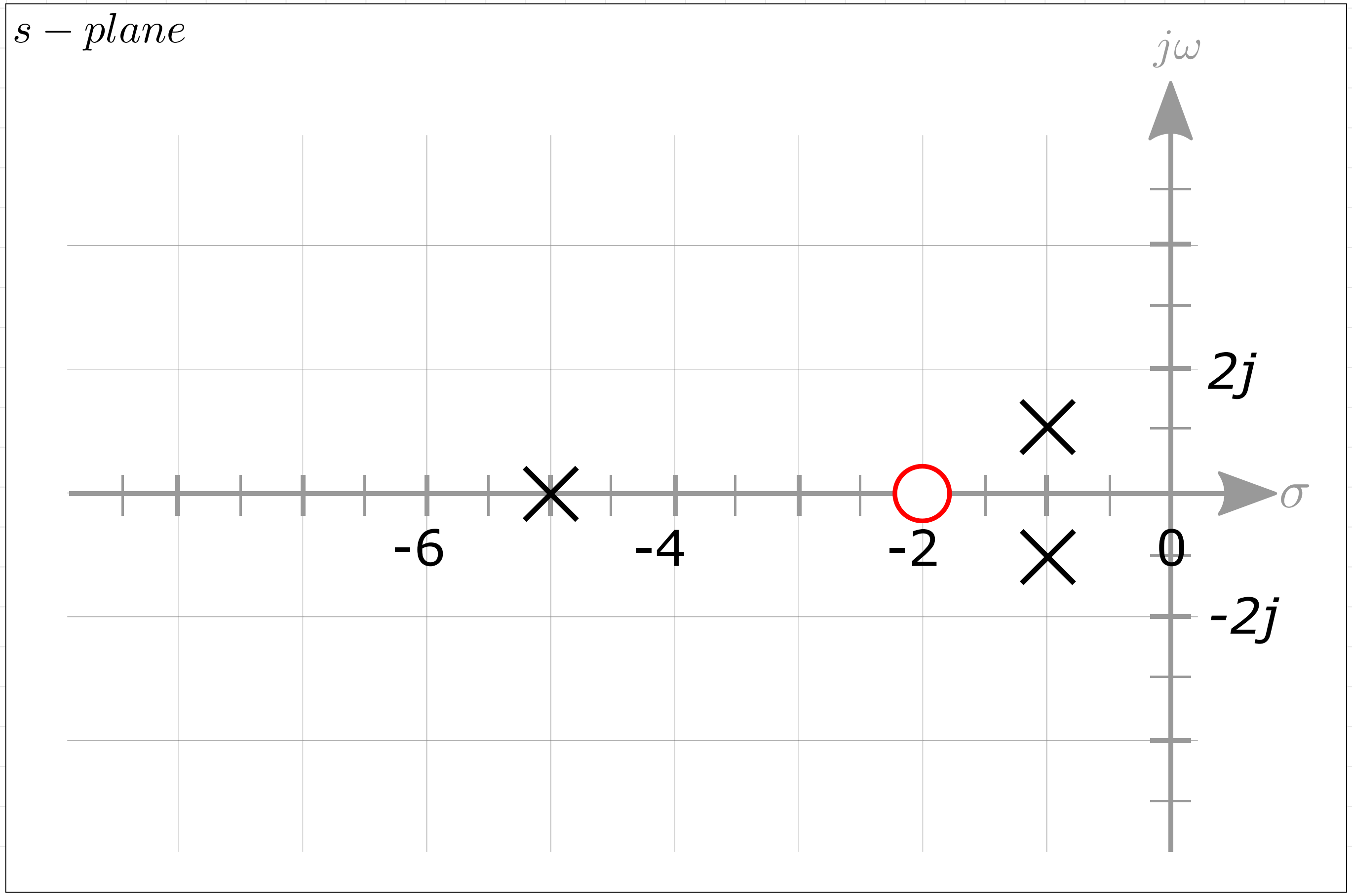
Derive the open-loop transfer function and sketch a refined root-locus for the feedback system, for which the open-loop poles and zeros are shown on the s-plane.

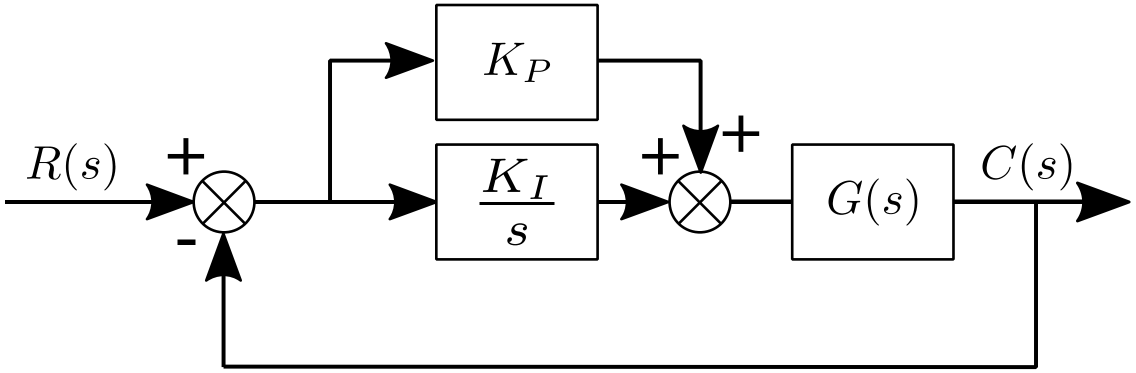


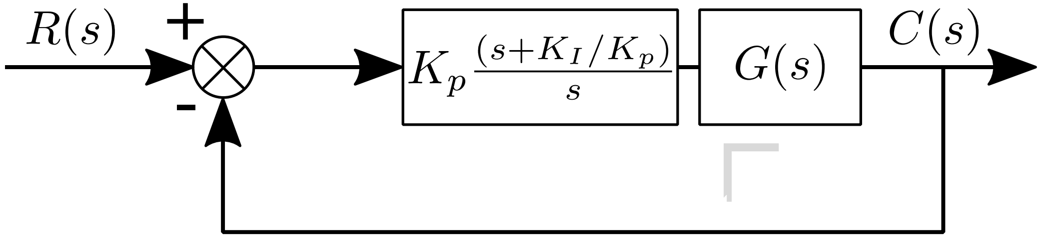
Derive the open-loop transfer function and sketch a refined root-locus for the feedback system, for which the open-loop poles and zeros are shown on the s-plane.



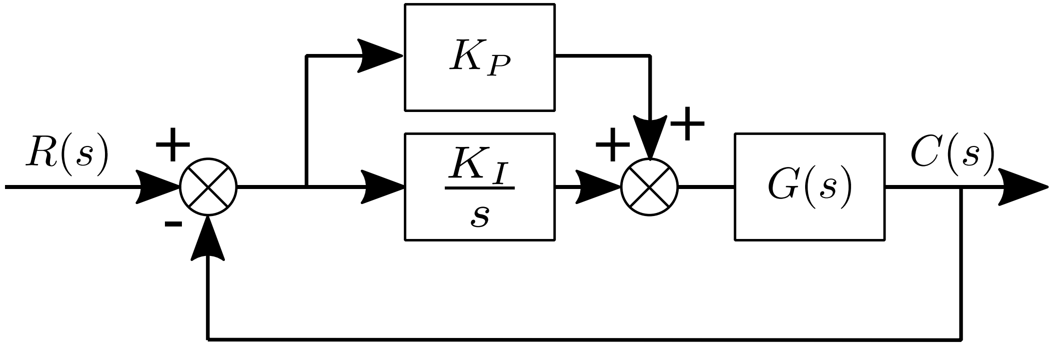
Generalized Root-Locus

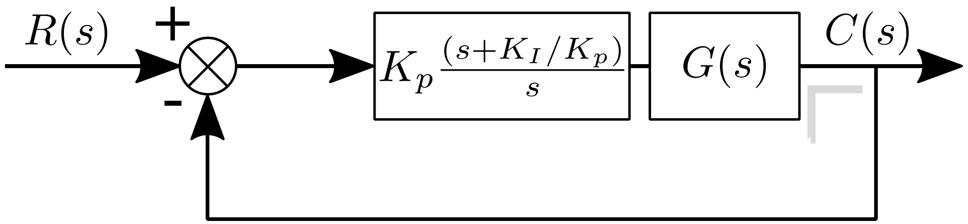
- The root-locus technique is not restricted to varying the gain K in a feedback system. It can be used to design for other parameters in a controller.
- Consider the case where we have a PI controller and want to plot the root-locus for varying location of the zero defined by $z = -\frac{K_I}{K_P}$, rather than varying the gain K_p
  - *In other words: Our design goal is to place the zero of the PI Controller (designing for the integral component), for a given value of the proportional gain* K_P



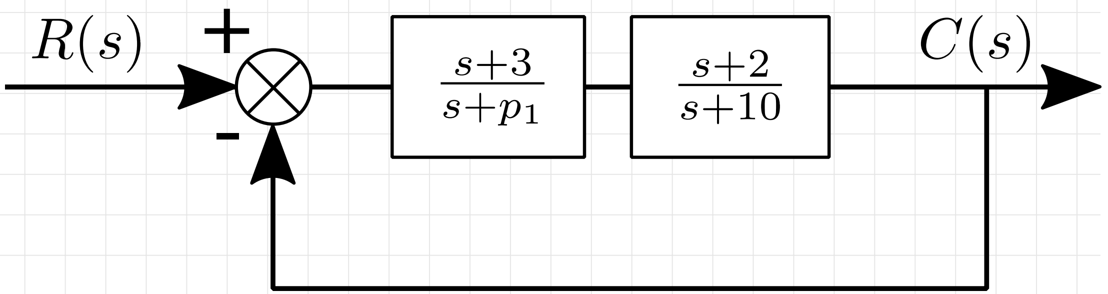


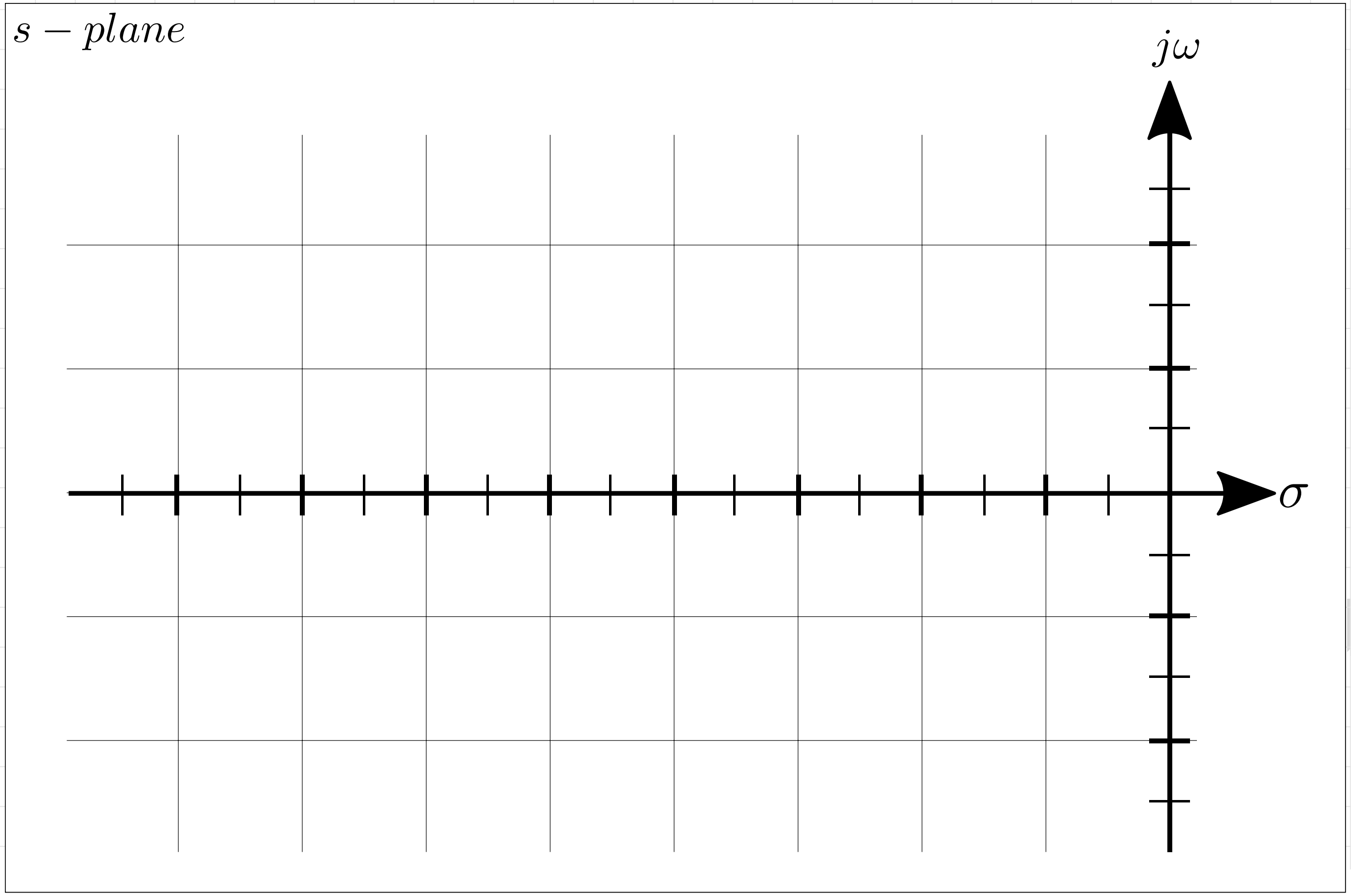
- Let $K_P = 1$ for simplicity, then the characteristic polynomial becomes:
- $1 + \frac{s+K_I}{s} G(s) = 0 \Rightarrow s + sG(s) + K_I G(s) = 0 \Rightarrow 1 + K_I \frac{G(s)}{s(1+G(s))} = 0$
  - *What we did is manipulate the characteristic poly into the unity feedback form.*
- The manipulated open-loop t.f. for which K_I (the zero location added by the PI controller) is then: $K_I \frac{G(s)}{s(1+G(s))}$
- We proceed to plot the root-locus, this time we get the closed-loop pole locations for varying values of K_I





Given the following controller, sketch the root-locus with p_1 as the controller design variable.





Practice Problems

Nise's 6^{th} Global Edition:

8-1, 8-2, 8-3,8-6,8-11,8-18,8-23

The root-locus sketching parts only, the design components will be covered in the following lectures.

Almost all problems from 8-1 to 8-23 are good practice problems for learning how to sketch the root-locus.