Kuwait University
College of Engineering and Petroleum

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ME417 CONTROL OF MECHANICAL SYSTEMS

PART II: CONTROLLER DESIGN VIA STATE-SPACE

LECTURE 1: STATE-SPACE REPRESENTATION

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- Objectives:
  - Introduction to modeling in the time domain
  - Treating the general state-space representation
  - Converting from state-space to transfer function and back
- Reading:
  - *Nise: 3.1-3.6*
- Practice Problems Included

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- But why not just use transfer functions?
- Transfer functions are powerful. They help us understand dynamic systems behavior and they have a good set of tools (root-locus, bode plots)
- However, they are limited to
    1. Linear systems
        - Real world systems are often nonlinear
    2. Time-invariant systems
        - Systems may change their properties over time
- High order systems produce mathematically complex transfer functions
- A transfer function can only relate one input to one output
- Multi-degree of freedom systems must be decoupled if T.F. representation is desired

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- The general state-space representation is

$$\dot{\boldsymbol{x}}(t) \in \mathcal{R}^{n\times 1} = f(\boldsymbol{x}, t, \boldsymbol{u}) = \boldsymbol{A}(\boldsymbol{x}, t) + \boldsymbol{B}(\boldsymbol{x}, t)$$

$$y(t) = \boldsymbol{C}(\boldsymbol{x}, t) + \boldsymbol{D}(\boldsymbol{x}, t)$$

$f(x, t, u)$ is called the system model, which is a function of the system state x, time t and the input to the system u

- Note that $f(x, t, u)$ correspond to the model function we use to simulate a general dynamic system numerically.
- The state-space representation can represent
  - Time-varying systems
  - Nonlinear dynamics
  - Multiple input and multiple outputs
  - Matrix representation for complex systems
    - Increasing system order increases matrix dimensions, rather than high order polynomial

- For linear time-invariant systems (the scope of this course) the representation is simplified to

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \mathbf{C}\boldsymbol{x} + \mathbf{D}\boldsymbol{u}$$

$\boldsymbol{x} \in \mathcal{R}^{n}$: State vector, $\dot{\boldsymbol{x}} \in \mathcal{R}^{n}$: derivative of state vector,

$\mathbf{A} \in \mathcal{R}^{n\times n}$: Constant System matrix, $\mathbf{B} \in \mathcal{R}^{n\times p}$: Constant Input matrix

$\boldsymbol{u} \in \mathcal{R}^{p}$: Input vector,

$\boldsymbol{y} \in \mathcal{R}^{m}$: Output vector, $\mathbf{C} \in \mathcal{R}^{m\times n}$: Constant Output matrix,

$\mathbf{D} \in \mathcal{R}^{n\times p}$: Constant Feedforward matrix

- $\in \mathcal{R}^{n\times n}$: denotes the matrix size is $n \times n$ and its values are real
- $\in \mathcal{R}^{n}$: denotes the vector size is n and its values are real

- n: Number of state variables, p: Number of inputs, m: Number of outputs

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- Matrix Addition/Subtraction

$$\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{B} \Rightarrow \underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -5 \\ 2 & 2 & 7 \end{bmatrix}}_{3x3} + \underbrace{\begin{bmatrix} -1 & 1 & 1 \\ 1 & 4 & 5 \\ 2 & -2 & 7 \end{bmatrix}}_{3x3} = \underbrace{\begin{bmatrix} 0 & -1 & 4 \\ 1 & 6 & 0 \\ 4 & 0 & 14 \end{bmatrix}}_{3x3}$$

- Matrix Scaling

$$c\mathbf{A} = \mathbf{B} \Rightarrow 3\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix}}_{3x3} = \underbrace{\begin{bmatrix} 3 & 6 & 3 \\ 9 & 6 & 9 \\ 12 & 9 & 6 \end{bmatrix}}_{3x3}$$

- Matrix Multiplication

$$\mathbf{A}x = y \Rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix}}_{3x3} \underbrace{\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}}_{3x1} = \underbrace{\begin{bmatrix} 13 \\ 23 \\ 28 \end{bmatrix}}_{3x1}, \qquad \mathbf{AB} = \mathbf{C} \Rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 2 & 1 & 2 \end{bmatrix}}_{3x3} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}}_{3x3} = \underbrace{\begin{bmatrix} 12 & 12 & 3 \\ 18 & 12 & 0 \\ 8 & 10 & 6 \end{bmatrix}}_{3x3}$$

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- Matrix Inverse

$$\mathbf{A} \in \mathcal{R}^{n\times n}, \quad \mathbf{A}^{-1} = \frac{adj(\mathbf{A})}{det(\mathbf{A})}, \quad \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

- Matrix Adjugate

$$for\ \mathbf{A} \in \mathcal{R}^{2\times 2}, \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{adj}(\mathbf{A}) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Matrix Determinant

$$for\ \mathbf{A} \in \mathcal{R}^{2\times 2}, \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(\mathbf{A}) = |\mathbf{A}| = ab - cd$$

- Matrix Eigenvalues

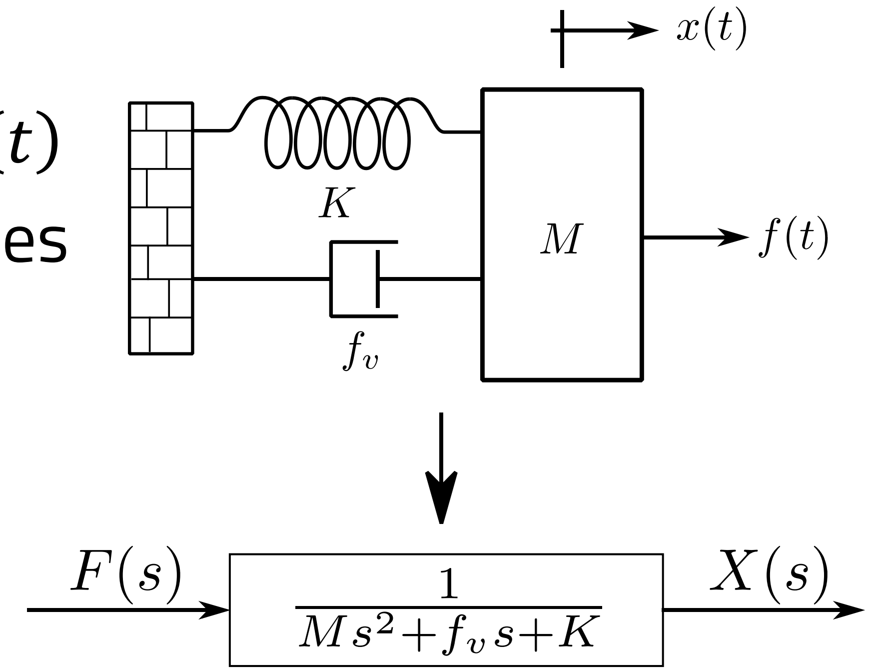
$\lambda_i \in \mathcal{C}^n$ are the special constants of a matrix $\mathbf{A} \in \mathcal{R}^{n\times n}$, called the eigenvalues

$$\mathbf{A}x = \lambda_i x, Ax - \lambda_i x = (\mathbf{A} - \lambda_i \mathbf{I})x = 0$$

The trivial solution is $x = \mathbf{0}$, while the nontrivial solution occurs if the determinant vanishes

$$\det(\mathbf{A} - \lambda_i \mathbf{I}) = 0,\ i = 1,2,\dots,n$$

- Consider the mass-spring-damper mechanical system
- The equation of motion is: $M\ddot{x} + f_v\dot{x} + Kx = f(t) = u(t)$
- This is a second-order system, requiring 2 state variables



  - Corresponding to the two derivatives

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

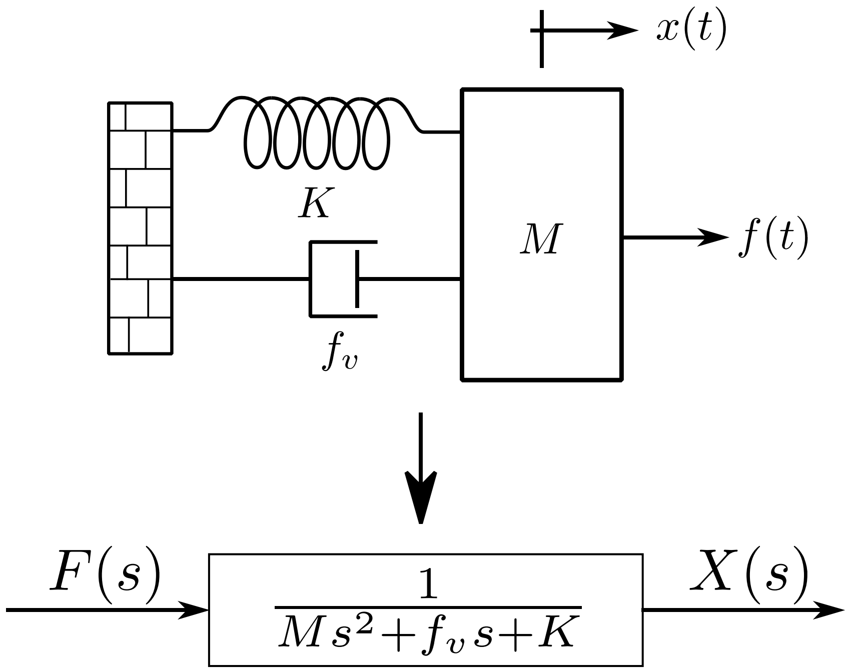
$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ (u(t) - f_v\dot{x} - Kx)/M \end{bmatrix} = \begin{bmatrix} x_2 \\ -f_v/Mx_2 - K/Mx_1 \end{bmatrix} + \begin{bmatrix} 0 \\ u(t)/M \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -K/M & -f_v/M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t) = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}$$

$$y = x = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u(t) = \mathbf{C}\boldsymbol{x} + \mathbf{D}\boldsymbol{u}$$

$$\mathbf{A} \in \mathcal{R}^{\mathbf{2}\times\mathbf{2}} = \begin{bmatrix} 0 & 1 \\ -K/M & -f_v/M \end{bmatrix}, \quad \mathbf{B} \in \mathcal{R}^{\mathbf{2}\times\mathbf{1}} = \begin{bmatrix} 0 \\ 1/M \end{bmatrix},$$

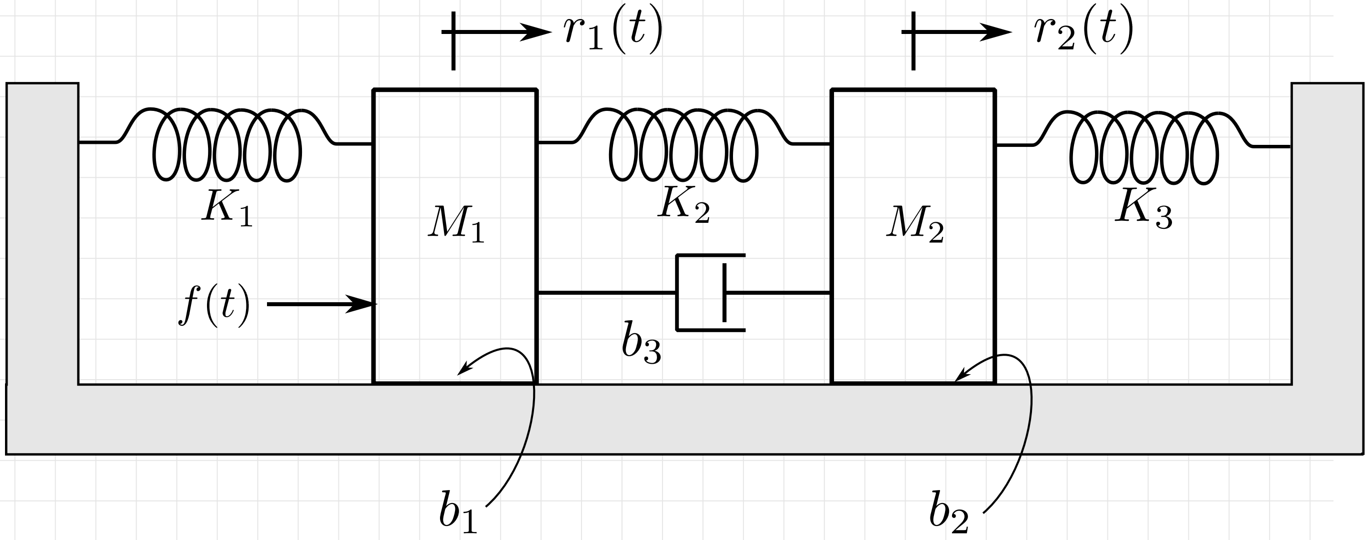
$$\mathbf{C} \in \mathcal{R}^{\mathbf{1}\times\mathbf{2}} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} \in \boldsymbol{\mathcal{R}} = \mathbf{0}$$



- In this course $\mathbf{D} = 0$

Derive the dynamic model of the system shown in state-space form

Example 1



- Given an n[th] order differential equation

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

- We can write it as n simultaneous first-order differential equations:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n-1}x_{n-1} + a_{1n}x_n + b_1 f(t)$$

$$\vdots$$

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in-1}x_{n-1} + a_{in}x_n + b_i f(t)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn-1}x_{n-1} + a_{nn}x_n + b_n f(t)$$

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- In state-space form, a general n^th^-order set of differential equations can be represented as:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} f(t)$$

- The output y depends on what we are measuring. If state x_1 is the output then:

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + 0$$

- Note that we use $y(t)$ or $c(t)$ interchangeably, to express the output. In state-space, it is common convention to use $y(t)$ for the output.

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- To convert a transfer function into state space, we first convert it to differential equation form
- Given: $G(s) = \frac{C(s)}{U(s)} = \frac{b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$, converting to diff. eq.
- Gives: $a_n \frac{d^n c(t)}{dt^n} + a_n \frac{d^n c(t)}{dt^n} + \cdots + a_1 \frac{dc(t)}{dt} + a_0 c(t) = b_0 u(t)$
- An n^th order diff. eq. gives n states

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [c \quad dc/dt \quad \ldots \quad d^{n-1}c/dt^{n-1} \quad d^n c/dt^n]^T$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & \ldots \\ \ldots & 0 & 1 & 0 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \ldots & \ldots & \ldots & 0 & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 & 1 \\ a_0/a_n & a_1/a_n & \ldots & \ldots & a_{n-2}/a_n & a_{n-1}/a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1/a_n \end{bmatrix} u(t)$$

$$y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + 0$$

- The state-space form retrieved from the transfer function, is called a phase variable form
  - Note the off-diagonal identity matrix, and how all the coefficients are grouped in the nth row

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & \dots \\ \dots & 0 & 1 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \dots & \dots & \dots & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & 0 & 1 \\ a_0/a_n & a_1/a_n & \dots & \dots & a_{n-2}/a_n & a_{n-1}/a_n \end{bmatrix}$$

- Unlike the state-space form retrieved from the equations of motion directly.
- Both the standard, and phase variable forms are valid representations

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Convert the following transfer function into state space form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+3)+5}$$

Example 2 B

Convert the following transfer function into state space form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{(s-3)}{s(s+9)}$$

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- Given a state space representation of an LTI system, there is an analytical solution to expressing the system in transfer function form
- Given

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

- Take the Laplace transform of both sides, we get

$$sX(s) = \boldsymbol{A}X(s) + \boldsymbol{B}U(s)$$
$$Y(s) = \boldsymbol{C}X(s) + \boldsymbol{D}U(s)$$

- Solving for $\boldsymbol{X}(s)$

$(s\boldsymbol{I} - \boldsymbol{A})X(s) = \boldsymbol{B}U(s) \quad \Rightarrow \quad X(s) = (s\mathbf{I} - \boldsymbol{A})^{-1}\boldsymbol{B}U(s)$, where $\mathbf{I}$ is the identity matrix

- Substituting $X(s)$ in $Y(s)$

$$Y(s) = \mathbf{C}\left((s\boldsymbol{I} - \mathbf{A})^{-1}\mathbf{B}U(s)\right) + \mathbf{D}U(s)$$

$$Y(s) = \left[\mathbf{C}\left((s\boldsymbol{I} - \mathbf{A})^{-1}\mathbf{B}\right) + \mathbf{D}\right]U(\boldsymbol{s})$$

- The term $\left[\mathbf{C}\left((s\boldsymbol{I} - \mathbf{A})^{-1}\mathbf{B}\right) + \mathbf{D}\right]$, is the *transfer function matrix*

$$G(s) = \frac{Y(s)}{U(s)} = \left[\mathbf{C}\left((s\boldsymbol{I} - \mathbf{A})^{-1}\mathbf{B}\right) + \mathbf{D}\right]$$

- The inverse term $(s\boldsymbol{I} - \mathbf{A})^{-1}$, can be computed as

$$(s\boldsymbol{I} - \mathbf{A})^{-1} = \frac{adj(\mathbf{sI} - \mathbf{A})}{det(\mathbf{sI} - \mathbf{A})}$$

Example 3

From the following system given in state-space form, find the transfer function

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 5 & 3 \\ 2 & 3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

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From the following system given in state-space form, represent the transfer function $G_2(s) = \frac{\Theta_2(s)}{M(s)}$ by the transfer function matrix, then compute using MATLAB

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 2 & 3 & -2 & -3 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}u(t), \quad \boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\boldsymbol{x}, \qquad \boldsymbol{x} = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

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- Note that the term $det(\boldsymbol{s}\mathbf{I} - \mathbf{A})$, is the denominator of the transfer function.
- In other words: it's the characteristic equation for the transfer function.
  - The roots of which are the poles of the system
- Note that the poles of the system are a function of the system matrix **A**
- So, stability can be evaluated by knowing the matrix **A**
- Remember that stability is defined in the context of the natural response (no input): $\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$
- Also note that the poles of the system are the eigenvalues λ_i of the matrix **A**

Is the following system stable?

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 5 & 3 \\ 8 & 3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

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Nise 6th Global Edition:

3-4, 3-6, 3-9, 3-14, 3-17, 3-19, 3-28

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